CONTENTS

CHAPTER 1: THE MODULUS FUNCTION

Absolute Valued Equations

Example 1

Solve:

a $|2x-1|=3$

b $|x-4|=2x+1$

Example 2

Solve the equation $|3x+4|=|x+5|$.

Example 3

Solve $|x+3|+|x+5|=10$.

Graphs of Absolute Valued Functions

Graphs of $y=|f(x)|$ where $f(x)$ is linear

Example 4

Sketch the graphs of the following on separate axes.

(i) $y=1-x$

(ii) $y=|1-x|$

(iii) $y=2+|1-x|$

Example 5

Sketch the graph of $y=\left|\frac{1}{2}x-1\right|$, showing the points where the graph meets the axes.

Use your graph to express $\left|\frac{1}{2}x-1\right|$ in an alternative form.

Sketch the following graphs.

a) $y=|2x-4|$

b) $y=2|x|-4$

Example 6

Describe fully the transformation (or combination of transformations) that maps the graph of $y=|x|$ onto each of these functions.

a $y=|x+1|+2$

b $y=|x-5|-2$

c $y=2-|x|$

d $y=|2x|-3$

e $y=1-|x+2|$

f $y=5-2|x|$

Absolute Valued Inequalities

Example 7

Solve the following.

(i) $|x+3| \leqslant 4$

(ii) $|2x-1| > 9$

(iii) $5-|x-2| > 1$

Example 8

Solve the inequality $|2x-1| \geqslant |3-x|$.

Example 9

Solve $2x < |x-3|$.

Example 10

Solve. (You may use either an algebraic method or a graphical method.)

a $|2x-5| \leqslant x-2$

b $|3+x| > 4-2x$

c $|x-7|-2x \leqslant 4$

Selected Past paper questions

Example 11

Given that a is a positive constant, solve the inequality $|x-3a| > |x-a|$.

[Cambridge International AS & A Level Mathematics 9709, Paper 3 Q1 November 2005]

Example 12

It is given that a is a positive constant.

(a) **(i)** Sketch on a single diagram the graphs of $y=|2x-3a|$ and $y=|2x+4a|$. [2]

(ii) State the coordinates of each of the points where each graph meets an axis. [1]

(b) Solve the inequality $|2x-3a| < |2x+4a|$. [3]

Specimen 2020 Paper 2 Question 3

Example 13

Solve the inequality $|3x-1| < |2x+5|$. [4]

November 2014/33 Question 1

Example 14

Find the set of values of x satisfying the inequality

$$|x + 2a| > 3|x - a|,$$

where a is a positive constant. [4]

June 2014/32 Question 1

HOMEWORK: THE MODULUS FUNCTION VARIANT 32

1	Solve the inequality $\lvert x-2\rvert < 3-2x$.	[4]
	Answer: $x < 1$.	J03/Q3
2	Solve the inequality $\lvert 2x+1\rvert < \lvert x\rvert$.	[4]
	Answer: $-1 < x < -\frac{1}{3}$.	J04/Q2
3	Given that a is a positive constant, solve the inequality $$\lvert x-3a\rvert > \lvert x-a\rvert.$$	[4]
	Answer: $x < 2a$.	N05/Q1
4	Solve the inequality $2x > \lvert x-1\rvert$.	[4]
	Answer: $x > \frac{1}{3}$.	J06/Q2
5	Solve the inequality $\lvert x-2\rvert > 3\lvert 2x+1\rvert$.	[4]
	Answer: $-1 < x < -\frac{1}{7}$.	J08/Q1
6	Solve the inequality $2\lvert x-3\rvert > \lvert 3x+1\rvert$.	[4]
	Answer: $-7 < x < 1$.	N10/32/Q1
7	Solve the inequality $\lvert 9-2x\rvert < 1$.	[3]
	Answer: $4 < x < 5$.	N02/Q1
8	Solve the inequality $\lvert x\rvert < \lvert 5+2x\rvert$.	[3]
	Answer: $x < -5,\ x > -\frac{5}{3}$.	J11/32/Q1
9	Solve the equation $\lvert x-2\rvert = \lvert \frac{1}{3}x\rvert$.	[3]
	Answer: $\frac{3}{2}$, 3	J13/32/Q1

10 Find the set of values of x satisfying the inequality

$$|x + 2a| > 3|x - a|,$$

where a is a positive constant. [4]

Answer: $\frac{1}{4}a < x < \frac{5}{2}a$ **J14/32/Q1**

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11 Find the set of values of x satisfying the inequality $3|x - 1| < |2x + 1|$. [4]

Answer: $\frac{2}{5} < x < 4$. **N12/32/Q1**

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12 Solve the inequality $|2x - 5| > 3|2x + 1|$. [4]

Answer: $-2 < x < \frac{1}{4}$ **N15/32/Q1**

HOMEWORK: THE MODULUS FUNCTION – VARIANTS 31 & 33

1 Solve the inequality $|x-2| > 2x-3$.

Answer: $nx < \frac{5}{3}$ 33/J15/2

2 Solve the inequality $|3x-1| < |2x+5|$. [4]

Answer: $-\frac{4}{5} < x < 6$ 33/N14/1

3 Solve the inequality $|4x+3| > |x|$. [4]

Answer: $x < -1$, $x > -\frac{3}{5}$ 33/J13/1

4 (i) Solve the equation $|4x-1| = |x-3|$. [3]

Answer: $-\frac{2}{3}$ and $\frac{4}{5}$ 31/J13/4

5 Find the set of values of x satisfying the inequality $3|x-1| < |2x+1|$. [4]

Answer: $\frac{2}{5} < x < 4$. 31/N12/1

6 Solve the inequality $2|x-3| > |3x+1|$. [4]

Answer: $-7 < x < 1$. 31/N10/1

7 Solve the inequality $|x-3| > 2|x+1|$. [4]

Answer: $-5 < x < \frac{1}{3}$. 33/J10/1

8 Solve the inequality $|x+3a| > 2|x-2a|$, where a is a positive constant. [4]

Answer: $\frac{a}{3} < x < 7a$. 31/J10/1

9 Find the set of values of x satisfying the inequality $2|2x-a| < |x+3a|$, where a is a positive constant. [4]

Answer: $-\frac{a}{5} < x < \frac{5a}{3}$ N18/33/Q1

10 **(i)** Solve the equation $2|x-1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

Answers: **(i)** -2, $\frac{2}{5}$ **(ii)** -0.569 **J16/31/Q1**

11 Solve the inequality $2|x-2| > |3x+1|$. [4]

Answer: $-5 < x < \frac{3}{5}$ **J16/33/Q1**

12 Solve the inequality $|2x+1| < 3|x-2|$. [4]

Answer: $x < 1$ and $x > 7$ **J17/31/Q1**

CHAPTER 2: POLYNOMIALS (REMAINDER FACTOR THEOREM)

Multiplying and Dividing Polynomials

Example 1

Multiply $(x^3 + 3x - 2)$ by $(x^2 - 2x - 4)$.

Example 2

Divide $x^3 - 5x^2 + x + 10$ by $(x - 2)$.

Example 4

Find the quotient and the remainder when $2x^3 + 3x^2 - 4x + 5$ is divided by $(x + 2)$.

Example 5

Find the quotient and the remainder when $x^4 + 2x^3 + 3x^2 + 7$ is divided by $(x^2 + x + 1)$.

Example 6

Find the remainder when $4x^3 - 7x - 1$ is divided by $(2x + 1)$.

The Remainder Theorem

When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

When a polynomial $f(x)$ is divided by $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$.

Example 7

Find the remainder when $4x^3 + x^2 - 3x + 7$ is divided by $(x + 2)$.

Example 8

When $16x^4 - ax^3 + 8x^2 - 4x - 1$ is divided by $(2x - 1)$, the remainder is 3. Find the value of a.

Example 9

$f(x) = 6x^3 + ax^2 + bx - 4$

When $f(x)$ is divided by $x - 1$, the remainder is 3.

When $f(x)$ is divided by $3x + 2$, the remainder is -2.

Find the value of a and the value of b.

The factor theorem

For any polynomial $f(x)$, if $f(a) = 0$ then the remainder when $f(x)$ is divided by $(x - a)$ is zero. Thus $(x - a)$ is a factor of $f(x)$.

For any polynomial $f(x)$, if $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

Example 10

Show that $(x - 3)$ is a factor of $x^5 - 3x^4 + x^3 - 4x - 15$.

Example 11

$(x - 2)$ is a factor of $x^3 - 3x^2 + ax - 10$. Evaluate the coefficient a.

Solving Cubic equations

Example 12

Solve $x^3 - 3x^2 - 4x + 12 = 0$.

Example 13

Solve the following equations.

a) $2x^3 + 7x^2 - 7x - 12 = 0$

Example 14

The cubic polynomial $ax^3 + bx^2 - 3x - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(x - 1)$ and $(x + 2)$ are factors of $p(x)$.

(i) Find the values of a and b.

(ii) When a and b have these values, find the other linear factor of $p(x)$.

[Cambridge International AS & A Level Mathematics 9709, Paper 2 Q4 June 2006]

Example 15

The polynomial $2x^3 + 7x^2 + ax + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 1)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x + 2)$ the remainder is 5. Find the values of a and b.

[Cambridge International AS & A Level Mathematics 9709, Paper 2 Q4 June 2008]

Example 16

The polynomial $2x^3 - x^2 + ax - 6$, where a is a constant, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$.

(i) Find the value of a.

(ii) When a has this value, factorise $p(x)$ completely.

[Cambridge International AS & A Level Mathematics 9709, Paper 2 Q2 November 2008]

Example 17

The polynomial $x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x - 1)$ the remainder is 4.

(i) Find the values of a and b.

(ii) When a and b have these values, find the other two linear factors of $p(x)$.

[Cambridge International AS & A Level Mathematics 9709, Paper 2 Q6 June 2009]

Example 18

The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$.

(i) Find the value of a.

(ii) When a has this value, find the quadratic factor of $p(x)$.

[Cambridge International AS & A Level Mathematics 9709, Paper 3 Q2 June 2007]

Example 19

Example 10.9 *Suppose x^2+x+1 is a factor of $p(x) = x^4 - x^3 - 9x^2 - 10x + a$,*

(i) find the value of a and the other quadratic factor.

(ii) solve the equation $p(x) < 0$, showing all your workings clearly.

HOMEWORK REMAINDER FACTOR THEOREM

1 The polynomial $x^4 - 2x^3 - 2x^2 + a$ is denoted by $f(x)$. It is given that $f(x)$ is divisible by $x^2 - 4x + 4$.

(i) Find the value of a. [3]

(ii) When a has this value, show that $f(x)$ is never negative. [4]

J03/Q4

Answer: (i) 8.

2 The polynomial $2x^3 + ax^2 - 4$ is denoted by $p(x)$. It is given that $(x - 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

When a has this value,

(ii) factorise $p(x)$, [2]

(iii) solve the inequality $p(x) > 0$, justifying your answer. [2]

Answers: (i) -3; (ii) $(x-2)(2x^2+x+2)$; (iii) $x > 2$. N04/Q3

3 The polynomial $x^4 + 5x + a$ is denoted by $p(x)$. It is given that $x^2 - x + 3$ is a factor of $p(x)$.

(i) Find the value of a and factorise $p(x)$ completely. [6]

(ii) Hence state the number of real roots of the equation $p(x) = 0$, justifying your answer. [2]

J05/Q5

Answers: (i) $a = -6$, $p(x) = (x^2 - x + 3)(x + 2)(x - 1)$; (ii) 2.

4 The polynomial $x^3 - 2x + a$, where a is a constant, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$.

(i) Find the value of a. [2]

(ii) When a has this value, find the quadratic factor of $p(x)$. [2]

Answers: (i) 4; (ii) $x^2 - 2x + 2$. J07/Q2

5 The polynomial $x^4 + 3x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $x^2 + x + 2$ is a factor of $p(x)$. Find the value of a and the other quadratic factor of $p(x)$. [4]

N07/Q2

Answers: $a = 4$; $x^2 - x + 2$.

6 The polynomial $4x^3 - 4x^2 + 3x + a$, where a is a constant, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $2x^2 - 3x + 3$.

(i) Find the value of a. [3]

(ii) When a has this value, solve the inequality $p(x) < 0$, justifying your answer. [3]

Answers: (i) 3; (ii) $x < -\frac{1}{2}$ N08/Q5

7 The polynomial $2x^3 + ax^2 + bx - 4$, where a and b are constants, is denoted by $\mathrm{p}(x)$. The result of differentiating $\mathrm{p}(x)$ with respect to x is denoted by $\mathrm{p}'(x)$. It is given that $(x+2)$ is a factor of $\mathrm{p}(x)$ and of $\mathrm{p}'(x)$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

Answers: **(i)** 7, 4; **(ii)** $(x+2)^2(2x-1)$. **N09/32/Q5**

8 The polynomial $2x^3 + 5x^2 + ax + b$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(2x+1)$ is a factor of $\mathrm{p}(x)$ and that when $\mathrm{p}(x)$ is divided by $(x+2)$ the remainder is 9.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

Answers: **(i)** -4, -3; **(ii)** $(2x+1)(x+3)(x-1)$. **J10/32/Q5**

9 The polynomial $ax^3 - 20x^2 + x + 3$, where a is a constant, is denoted by $\mathrm{p}(x)$. It is given that $(3x+1)$ is a factor of $\mathrm{p}(x)$.

(i) Find the value of a. [3]

(ii) When a has this value, factorise $\mathrm{p}(x)$ completely. [3]

Answer: **(i)** 12; **(ii)** $(3x+1)(2x-1)(2x-3)$ **J13/32/Q4**

10 The polynomial $x^4 + 3x^3 + ax + 3$ is denoted by $\mathrm{p}(x)$. It is given that $\mathrm{p}(x)$ is divisible by $x^2 - x + 1$.

(i) Find the value of a. [4]

(ii) When a has this value, find the real roots of the equation $\mathrm{p}(x) = 0$. [2]

Answers: **(i)** $a = 1$; **(ii)** $x = -1$, $x = -3$ **N11/32/Q3**

11 The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(3x+1)$ is a factor of $\mathrm{p}(x)$, and that when $\mathrm{p}(x)$ is divided by $(x-2)$ the remainder is 21. Find the values of a and b. [5]

Answer: $a = 12$, $b = -20$ **N14/32/Q3**

12 The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $(x+1)$ is a factor of $\mathrm{p}(x)$ and that when $\mathrm{p}(x)$ is divided by $(2x+1)$ the remainder is 1.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $\mathrm{p}(x)$ completely. [3]

Answers: **(i)** $a = 6$, $b = -3$ **(ii)** $(x+1)(4x+1)(2x-1)$ **N15/32/Q6**

REMAINDER FACTOR THEOREM– VARIANTS 31 & 33

1 The polynomial $4x^3 + ax^2 + bx - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 1)$ and $(x + 2)$ are factors of $p(x)$.

(i) Find the values of a and b. [4]

(ii) When a and b have these values, find the remainder when $p(x)$ is divided by $(x^2 + 1)$. [3]

Answers: (i) $a = 11$, $b = 5$ (ii) $x - 13$ 33/N14/3

2 The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(3x + 1)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x - 2)$ the remainder is 21. Find the values of a and b. [5]

Answer: $a = 12$, $b = -20$ 31/N14/3

3 It is given that $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$.

(i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. [3]

(ii) By first using the factor theorem, factorise $16x^3 - 24x^2 - 15x - 2$ completely. [4]

(iii) Hence solve the equation $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$. [1]

Answer: (ii) $(x-2)(4x+1)^2$ (iii) $x = 2$ 31/J14/6

4 The polynomial $f(x)$ is defined by

$$f(x) = x^3 + ax^2 - ax + 14,$$

where a is a constant. It is given that $(x + 2)$ is a factor of $f(x)$.

(i) Find the value of a. [2]

(ii) Show that, when a has this value, the equation $f(x) = 0$ has only one real root. [3]

Answer: (i) $a = -1$ 33/N13/3

5 The polynomial $8x^3 + ax^2 + bx + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(2x - 1)$ the remainder is 1.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the remainder when $p(x)$ is divided by $2x^2 - 1$. [3]

Answer: $a = -10$ $b = -1$ *Answer*: $3x - 2$ 33/J13/5

6 Find the quotient and remainder when $2x^2$ is divided by $x + 2$. [3]

Answer: $2x - 4$, 8 31/J13/1

7 The polynomial $p(x)$ is defined by

$$p(x) = x^3 - 3ax + 4a,$$

where a is a constant.

(i) Given that $(x-2)$ is a factor of $p(x)$, find the value of a. [2]

(ii) When a has this value,

(a) factorise $p(x)$ completely, [3]

(b) find all the roots of the equation $p(x^2) = 0$. [2]

Answers: (i) 4 (ii) (a) $(x+4)(x-2)^2$ (b) $+\sqrt{2}, -\sqrt{2}, i2, -i2$ 31/J12/3

8 The polynomial $p(x)$ is defined by

$$p(x) = ax^3 - x^2 + 4x - a,$$

where a is a constant. It is given that $(2x-1)$ is a factor of $p(x)$.

(i) Find the value of a and hence factorise $p(x)$. [4]

(ii) When a has the value found in part (i), express $\frac{8x-13}{p(x)}$ in partial fractions. [5]

Answers: (i) 2, $(2x-1)(x^2+2)$; (ii) $\frac{-4}{2x-1} + \frac{2x+5}{x^2+2}$ 33/N11/7

9 The polynomial $ax^3 + bx^2 + 5x - 2$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x-1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x-2)$ the remainder is 12.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the quadratic factor of $p(x)$. [2]

Answers: (i) 2, −3; (ii) $x^2 - x + 2$. 33/J11/5

10 The polynomial $f(x)$ is defined by

$$f(x) = 12x^3 + 25x^2 - 4x - 12.$$

(i) Show that $f(-2) = 0$ and factorise $f(x)$ completely. [4]

(ii) Given that

$$12 \times 27^y + 25 \times 9^y - 4 \times 3^y - 12 = 0,$$

state the value of 3^y and hence find y correct to 3 significant figures. [3]

Answers: (i) $(x+2)(4x+3)(3x-2)$; (ii) −0.369. 31/J11/4

11 The polynomial $4x^4 + ax^2 + 11x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $x^2 - x + 2$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the real roots of the equation $p(x) = 0$. [2]

Answers: $a = 1$ $b = -6$ N16/33/Q4

Answers: real roots ½ and − 3/2 imaginary roots (1 ± i√7)/2 or discriminant = − 7

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12 The polynomial $x^4 + 2x^3 + ax + b$, where a and b are constants, is divisible by $x^2 - x + 1$. Find the values of a and b. [5]

Answer: $a = 1$, $b = 2$ J18/31/Q4

CHAPTER 3: BINOMIAL EXPANSION OF $(a+b)^n$

Example 1

Expand $(1-x)^{-2}$ as a series of ascending powers of x up to and including the term in x^3, stating the set of values of x for which the expansion is valid.

Example 2

Find the first three terms in the expansion of $\sqrt{(4-x)}$ in ascending powers of x.

Example 3

Find the first four terms in the expansion of $(1+2x^2)^{-4}$ and state the range of values of x for which the expansion is valid.

Example 4

Expand $\dfrac{1}{\sqrt{(4+x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

Example 5

Find the first four terms in the expansion of $\dfrac{1}{(2+x^2)}$ in ascending powers of x.

Example 6

Expand $(2+3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

[Cambridge International AS & A Level Mathematics 9709, Paper 3 Q1 June 2007]

Example 7

Expand $(1+x)\sqrt{(1-2x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

[Cambridge International AS & A Level Mathematics 9709, Paper 3 Q2 November 2008]

Example 8

Find a and b such that

$$\frac{1}{(1-2x)(1+3x)} \approx a + bx$$

and state the values of x for which the expansions you use are valid.

Example 9

When $(1+2x)(1+ax)^{\frac{2}{3}}$, where a is a constant, is expanded in ascending powers of x, the coefficient of the term in x is zero.

(i) Find the value of a.

(ii) When a has this value, find the term in x^3 in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient.

[Cambridge International AS & A Level Mathematics 9709, Paper 3 Q5 June 2009]

HOMEWORK: BINOMIAL EXPANSION VARIANT 32

1 Expand $(2 + x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients. [4]

N03/Q2

Answer: $\frac{1}{4} - \frac{1}{4}x^2 + \frac{3}{16}x^4$.

2 Expand $\dfrac{1}{(2+x)^3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

N04/Q1

Answer: $\frac{1}{8} - \frac{3}{16}x + \frac{3}{16}x^2$.

3 Expand $(1 + 4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

J05/Q1

Answer: $1 - 2x + 6x^2 - 20x^3$.

4 **(i)** Simplify $\left(\sqrt{(1+x)} + \sqrt{(1-x)}\right)\left(\sqrt{(1+x)} - \sqrt{(1-x)}\right)$, showing your working, and deduce that

$$\frac{1}{\sqrt{(1+x)} + \sqrt{(1-x)}} = \frac{\sqrt{(1+x)} - \sqrt{(1-x)}}{2x}.$$ [2]

(ii) Using this result, or otherwise, obtain the expansion of

$$\frac{1}{\sqrt{(1+x)} + \sqrt{(1-x)}}$$

in ascending powers of x, up to and including the term in x^2. [4]

N06/Q5

Answer: (ii) $\frac{1}{2} + \frac{1}{16}x^2$

5 Expand $(2 + 3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

J07/Q1

Answer: $\frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2$.

6 Expand $(1 + x)\sqrt{(1 - 2x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

N08/Q2

Answer: $1 - \frac{3}{2}x^2$

7 When $(1+2x)(1+ax)^{\frac{2}{3}}$, where a is a constant, is expanded in ascending powers of x, the coefficient of the term in x is zero.

(i) Find the value of a. [3]

(ii) When a has this value, find the term in x^3 in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient. [4]

Answers: (i) -3; (ii) $-\frac{10}{3}x^3$ J09/Q5

8 Expand $\sqrt{\left(\frac{1-x}{1+x}\right)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [5]

Answer: $1-x+\frac{1}{2}x^2$ J12/32/Q3

9 When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascending powers of x, the coefficients of x and x^3 are equal.

(i) Find the exact value of a. [4]

(ii) When a has this value, obtain the expansion up to and including the term in x^2, simplifying the coefficients. [3]

Answers: (i) $\frac{1}{\sqrt{2}}$; (ii) $1-\sqrt{2}x+\frac{3}{2}x^2$. N12/32/Q4

10 Given that $\sqrt[3]{(1+9x)} \approx 1+3x+ax^2+bx^3$ for small values of x, find the values of the coefficients a and b. [3]

Answer: $a = -9$, $b = 45$ N15/33/Q2

BINOMIAL EXPANSION– VARIANTS 31 & 33

1 Show that, for small values of x^2,

$$(1-2x^2)^{-2}-(1+6x^2)^{\frac{2}{3}} \approx kx^4,$$

where the value of the constant k is to be determined.

Answer: 16 — 31/J15/3

3 Expand $(1+3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

Answer: $1-x+2x^2-\frac{14}{3}x^3$ — 33/J14/2

4 Expand $\dfrac{1+3x}{\sqrt{(1+2x)}}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [4]

Answer: $1+2x-\frac{3}{2}x^2$ — 31/J13/2

5 When $(1+ax)^{-2}$, where a is a positive constant, is expanded in ascending powers of x, the coefficients of x and x^3 are equal.

(i) Find the exact value of a. [4]

(ii) When a has this value, obtain the expansion up to and including the term in x^2, simplifying the coefficients. [3]

Answers: (i) $\frac{1}{\sqrt{2}}$; (ii) $1-\sqrt{2}x+\frac{3}{2}x^2$. — 31/N12/4

6 Expand $\dfrac{1}{\sqrt{(4+3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

Answer: $\frac{1}{2}-\frac{3}{16}x+\frac{27}{256}x^2$ — 33/J12/1

7 **(i)** Expand $\dfrac{1}{\sqrt{(1-4x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

(ii) Hence find the coefficient of x^2 in the expansion of $\dfrac{1+2x}{\sqrt{(4-16x)}}$. [2]

Answers: (i) $1+2x+6x^2$ (ii) 5 — 31/J12/2

8 Expand $\dfrac{16}{(2+x)^2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

Answer: $4-4x+3x^2$ — 33/N11/1

9 Expand $\sqrt[3]{(1-6x)}$ in ascending powers of x up to and including the term in x^3, simplifying the coefficients. [4]

Answer: $1-2x-4x^2-\frac{40}{3}x^3$. 31/J11/1

10 Expand $(1+2x)^{-3}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

Answer: $1-6x+24x^2$. 33/N10/1

11 Expand $\frac{1}{\sqrt[3]{(1+6x)}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

Answer: $1-2x+8x^2-\frac{112}{3}x^3$ J17/31/Q2

12 Expand $(3+2x)^{-3}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [4]

Answer: $\frac{1}{27}-\frac{2}{27}x+\frac{8}{81}x^2$ J17/33/Q2

13 Expand $\frac{4}{\sqrt{(4-3x)}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

Answer: $2+\frac{3}{4}x+\frac{27}{64}x^2$ J18/33/Q1

CHAPTER 4: PARTIAL FRACTIONS

Type 1: Proper fraction where the denominator has distinct linear factors

$$\frac{1}{(x+p)(x+q)} \equiv \frac{A}{(x+p)} + \frac{B}{(x+q)}$$

$$\frac{px+q}{(ax+b)(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d}$$

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

Example 1

Express $\frac{x+3}{(x-2)(x+1)}$ in partial fractions.

Example 2

Express $\frac{2x-13}{(2x+1)(x-3)}$ in partial fractions.

Example 3

Express $\frac{x}{1-x-2x^2}$ in partial fractions.

Type 2: Proper fraction with repeated linear factor in the denominator

$$\frac{1}{(x+p)^2} \equiv \frac{A}{(x+p)} + \frac{B}{(x+p)^2}$$

$$\frac{px+q}{(ax+b)^2} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

Example 4

Express $\frac{1-9x-8x^2}{(x-2)(2x+3)^2}$ in partial fractions.

Example 5

Express $\dfrac{2x-11}{(x-4)^2}$ in partial fractions.

Example 6

Express $\dfrac{2x}{(x+3)^2}$ in partial fractions.

Type 3: Proper fraction with a quadratic factor in the denominator that cannot be factorized

$$\frac{1}{(x+p)(x^2+q)} \equiv \frac{A}{x+p} + \frac{Bx+C}{x^2+q}$$

$$\frac{px+q}{(ax+b)(cx^2+d)} \equiv \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}$$

Example 7

Express $\dfrac{2x^2-x+6}{(x+1)(x^2+2)}$ in partial fractions.

Example 8

Express $\dfrac{5x^2-5x+4}{(2x-1)(3-x^2)}$ in partial fractions.

Example 9

Express $\dfrac{8+2x-x^2}{(x-1)(x^2+2)}$ in partial fractions.

Type 4: Improper fractions

They can be split into partial fractions by first doing long division, and then splitting the remainder into partial fractions using one of the techniques discussed in types 1, 2, and 3:

$$\frac{x^2}{(x+p)(x+q)} \equiv A + \frac{Bx+C}{(x+p)(x+q)}$$

Example 10

Express $\dfrac{x^2 - x + 5}{(x-1)(x+2)}$ in partial fractions.

Example 11

Find the values of the constants A, B, C, and D such that

$$\frac{3x^3 + 2x^2 + 6x + 4}{x^2(x+1)} = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x+1}$$

Example 12

Express $\dfrac{7x^2 - 3x + 2}{x(x^2+1)}$ in partial fractions. [5]

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Partial Fractions with binomial Expansions

Example 13

a) Express $\dfrac{1}{(x+1)(x-1)^2}$ in partial fractions.

b) Hence obtain the expansion of $\dfrac{1}{(x+1)(x-1)^2}$ in ascending powers of x, up to and including the term in x^2.

Example 14

Given that $f(x) = \dfrac{10x+1}{(1-2x)(1+x)}$, express $f(x)$ in partial fractions and hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. State the range of values of x for which the expansion is valid.

Example 15

Let $f(x) = \dfrac{4x^2+12}{(x+1)(x-3)^2}$.

i Express $f(x)$ in partial fractions. **[5]**

ii Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. **[5]**

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Example 16

Let $f(x) = \dfrac{2x^2-7x-1}{(x-2)(x^2+3)}$.

i Express $f(x)$ in partial fractions. **[5]**

ii Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. **[5]**

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Example 17

Let $f(x) = \dfrac{3x}{(1+x)(1+2x^2)}$.

i Express $f(x)$ in partial fractions. **[5]**

ii Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. **[5]**

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PARTIAL FRACTIONS VARIANT 32

1

Let $f(x) = \dfrac{x^3 - x - 2}{(x-1)(x^2+1)}$.

(i) Express $f(x)$ in the form

$$A + \frac{B}{x-1} + \frac{Cx+D}{x^2+1},$$

where A, B, C and D are constants. [5]

Answer: (i) $1 - \dfrac{1}{x-1} + \dfrac{2x}{x^2+1}$. N03/Q8

2

An appropriate form for expressing $\dfrac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1} + \frac{B}{x-2},$$

where A and B are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\dfrac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$. [2]

N04/Q8

Answer: (a)(i) $\dfrac{A}{x+4} + \dfrac{Bx+C}{x^2+3}$; (ii) $\dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ or $\dfrac{A}{x-2} + \dfrac{Bx+C}{(x+2)^2}$.

3

Let $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

Answer: (i) $\dfrac{2}{2x+1} - \dfrac{1}{x+1} + \dfrac{3}{(x+1)^2}$ N06/Q8

4

Let $f(x) \equiv \dfrac{x^2+3x+3}{(x+1)(x+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

Answer: (i) $1 + \dfrac{1}{2(x+1)} - \dfrac{3}{2(x+3)}$. J08/Q7

5 (i) Express $\dfrac{100}{x^2(10-x)}$ in partial fractions. [4]

Answers: (i) $\dfrac{1}{x}+\dfrac{10}{x^2}+\dfrac{1}{10-x}$; J09/Q8

6 (i) Find the values of the constants A, B, C and D such that

$$\frac{2x^3-1}{x^2(2x-1)} \equiv A+\frac{B}{x}+\frac{C}{x^2}+\frac{D}{2x-1}.$$ [5]

Answers: (i) 1, 2, 1, −3. J10/32/Q10

7 Let $f(x)=\dfrac{9x^2+4}{(2x+1)(x-2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that, when x is sufficiently small for x^3 and higher powers to be neglected,

$$f(x)=1-x+5x^2.$$ [4]

Answer: (i) $\dfrac{1}{2x+1}+\dfrac{4}{x-2}+\dfrac{8}{(x-2)^2}$ or $\dfrac{1}{2x+1}+\dfrac{4x}{(x-2)^2}$. J03/Q6

8 Let $f(x)=\dfrac{x^2+7x-6}{(x-1)(x-2)(x+1)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x)=-3+2x-\tfrac{3}{2}x^2+\tfrac{11}{4}x^3.$$ [5]

Answer: (i) $-\dfrac{1}{x-1}+\dfrac{4}{x-2}-\dfrac{2}{x+1}$. J04/Q9

9 (i) Express $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in ascending powers of x, up to and including the term in x^3. [5]

Answers: (i) $\dfrac{2}{2+x}+\dfrac{x-1}{x^2+1}$; (ii) $\dfrac{1}{2}x+\dfrac{5}{4}x^2-\dfrac{9}{8}x^3$. N05/Q9

10

(i) Express $\dfrac{10}{(2-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence, given that $|x| < 1$, obtain the expansion of $\dfrac{10}{(2-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [5]

Answers: (i) $\dfrac{2}{2-x}+\dfrac{2x+4}{1+x^2}$; (ii) $5+\dfrac{5}{2}x-\dfrac{15}{4}x^2-\dfrac{15}{8}x^3$. **J06/Q9**

11

(i) Express $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: (i) $\dfrac{1}{1-x}+\dfrac{2}{1+2x}-\dfrac{4}{2+x}$; (ii) $1-2x+\dfrac{17}{2}x^2$ **N07/Q9**

12

(i) Express $\dfrac{1+x}{(1-x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{1+x}{(1-x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: (i) $\dfrac{2}{3(1-x)}+\dfrac{2x-1}{3(2+x^2)}$; (ii) $\dfrac{1}{2}+x+\dfrac{3}{4}x^2$. **N09/32/Q8**

13

Let $f(x) = \dfrac{3x}{(1+x)(1+2x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

Answers: (i) $-\dfrac{1}{1+x}+\dfrac{2x+1}{1+2x^2}$; (ii) $3x-3x^2-3x^3$. **N10/32/Q8**

14 Let $f(x) = \frac{6+7x}{(2-x)(1+x^2)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when x is sufficiently small for x^4 and higher powers to be neglected,

$$f(x) = 3 + 5x - \tfrac{1}{2}x^2 - \tfrac{15}{4}x^3.$$ [5]

Answer: (i) $\frac{4}{2-x} + \frac{4x+1}{1+x^2}$. N02/Q6

15 (i) Express $\frac{5x-x^2}{(1+x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\frac{5x-x^2}{(1+x)(2+x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

Answers: (i) $-\frac{2}{1+x} + \frac{x+4}{2+x^2}$; (ii) $\frac{5}{2}x - 3x^2 + \frac{7}{4}x^3$. J11/32/Q8

16 Let $f(x) = \frac{5x^2+x+6}{(3-2x)(x^2+4)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: (i) $\frac{3}{3-2x} - \frac{x+2}{x^2+4}$; (ii) $\frac{1}{2} + \frac{5}{12}x + \frac{41}{72}x^2$ J15/32/Q8

17 Let $f(x) = \frac{2x^2-7x-1}{(x-2)(x^2+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answer: (i) $-\frac{1}{x-2} + \frac{3x-1}{x^2+3}$; (ii) $\frac{1}{6} + \frac{5}{4}x + \frac{17}{72}x^2$ N13/32/Q7

28 Let $f(x) = \frac{x^2-8x+9}{(1-x)(2-x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: **(i)** $\frac{2}{(1-x)}-\frac{1}{(2-x)}+\frac{3}{(2-x)^2}$, or $\frac{2}{(1-x)}+\frac{x+1}{(2-x)^2}$ **(ii)** $\frac{9}{4}+\frac{5}{2}x+\frac{39}{16}x^2$ **N14/32/Q9**

PARTIAL FRACTIONS– VARIANTS 31 & 33

1 Let $f(x) = \dfrac{11x+7}{(2x-1)(x+2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

33/J15/10

Answers: (i) $\dfrac{2}{2x-1} + \dfrac{-1}{x+2} + \dfrac{3}{(x+2)^2}$

---

2 Let $f(x) = \dfrac{x^2-8x+9}{(1-x)(2-x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

31/N14/9

Answers: (i) $\dfrac{2}{(1-x)} - \dfrac{1}{(2-x)} + \dfrac{3}{(2-x)^2}$, or $\dfrac{2}{(1-x)} + \dfrac{x+1}{(2-x)^2}$ (ii) $\dfrac{9}{4} + \dfrac{5}{2}x + \dfrac{39}{16}x^2$

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3 Let $f(x) = \dfrac{6+6x}{(2-x)(2+x^2)}$.

Answer: (i) $A = 3$, $B = 3$, $C = 0$

33/J14/8

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4 (i) Express $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

31/J14/9

Answers: (i) $\dfrac{1}{3-x} + \dfrac{3}{2(1+2x)} - \dfrac{1}{2(1+2x)^2}$ or $\dfrac{1}{3-x} + \dfrac{3x+1}{(1+2x)^2}$ (ii) $\dfrac{4}{3} - \dfrac{8}{9}x + \dfrac{1}{27}x^2$

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5 (i) Express $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in partial fractions. [5]

(ii) Hence expand $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [5]

Answer: (i) $\frac{-1}{1+x}+\frac{3}{(1+x)^2}+\frac{4}{2-3x}$ (ii) $4-2x+\frac{25}{2}x^2$ 33/N13/8

6 (i) Express $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{9-7x+8x^2}{(3-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3. [5]

Answers: (i) $\frac{6}{3-x}+\frac{-2x+1}{1+x^2}$; (ii) $3-\frac{4}{3}x-\frac{7}{9}x^2+\frac{56}{27}x^3$. 33/N12/9

7 Let $f(x)=\dfrac{4x^2-7x-1}{(x+1)(2x-3)}$.

(i) Express $f(x)$ in partial fractions. [5]

Answer: (ii) $2-\frac{2}{x+1}-\frac{1}{2x-3}$ 33/J12/8

8 Let $f(x)=\dfrac{12+8x-x^2}{(2-x)(4+x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{2-x}+\dfrac{Bx+C}{4+x^2}$. [4]

Answer: (i) $\frac{3}{2-x}+\frac{4x}{4+x^2}$ 31/N11/8

9 Let $f(x)=\dfrac{3x}{(1+x)(1+2x^2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

Answers: (i) $-\frac{1}{1+x}+\frac{2x+1}{1+2x^2}$; (ii) $3x-3x^2-3x^3$. 31/N10/8

10 (i) Express $\dfrac{4+5x-x^2}{(1-2x)(2+x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4+5x-x^2}{(1-2x)(2+x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: (i) $\dfrac{1}{1-2x}+\dfrac{1}{2+x}-\dfrac{2}{(2+x)^2}$; (ii) $1+\dfrac{9}{4}x+\dfrac{15}{4}x^2$. 33/J10/9

---

11 (i) Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part (i), show that

Answer: (i) $\dfrac{1}{x+1}-\dfrac{1}{x+3}$. 31/J10/8

---

12 Let $\mathrm{f}(x)=\dfrac{3x^2+x+6}{(x+2)(x^2+4)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answer: $\dfrac{2}{x+2}+\dfrac{x-1}{x^2+4}$ Answer: $\dfrac{3}{4}-\dfrac{1}{4}x+\dfrac{5}{16}x^2$ N16/33/Q8

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13 Let $\mathrm{f}(x)=\dfrac{4x^2+12}{(x+1)(x-3)^2}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answers: (i) $\dfrac{1}{x+1}+\dfrac{3}{x-3}+\dfrac{12}{(x-3)^2}$, (ii) $\dfrac{4}{3}-\dfrac{4}{9}x+\dfrac{4}{3}x^2$ J16/31/Q8

---

14 Let $\mathrm{f}(x)=\dfrac{10x-2x^2}{(x+3)(x-1)^2}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answer: (i) $\dfrac{-3}{x+3}+\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$ or $\dfrac{-3}{x+3}+\dfrac{x+1}{(x-1)^2}$; (ii) $\dfrac{10}{3}x+\dfrac{44}{9}x^2$ J16/33/Q10

15 Let $f(x) = \dfrac{12x^2 + 4x - 1}{(x-1)(3x+2)}$.

(i) Express $f(x)$ in partial fractions. [5]

15

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

Answer: **(i)** $4 + \dfrac{3}{x-1} - \dfrac{1}{3x+2}$ **(ii)** $\dfrac{1}{2} - \dfrac{9}{4}x - \dfrac{33}{8}x^2$ **J18/31/Q9**

CHAPTER 5: EXPONENTS AND LOGARITHMS

Exponents

Example 1

Find the values of (a) $100^{\frac{3}{2}}$ *(b)* $32^{-\frac{2}{5}}$.

Example 2

Show that (a) $3^{2x-1} = \frac{9^x}{3}$, *(b)* $2^{x-1} \times 8^{x+1} = 4^{2x+1}$

Example 3

Solve the equations (a) $3^x = 81$, *(b)* $8^x = 0.25$.

Example 4

Solve the equation $2^{2x+3} + 1 = 9 \times 2^x$.

Example 5

Solve the equation $2^{2x+1} + 15 \times 2^x - 8 = 0$.

Example 6

Solve the equation $2^x + 2^{1-x} = 3$.

Example 7

Solve the simultaneous equations

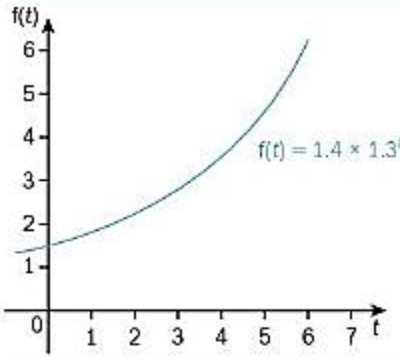
$$3^x \times 9^y = 1 \qquad \text{(i)}$$

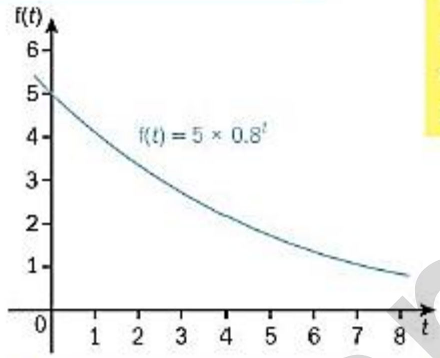
and

$$2^{2x} \times 4^y = \frac{1}{8} \qquad \text{(ii)}$$

The Exponential Function

A function $f(t) = a \times r^t$, where t is real,
gives exponential growth ($r > 1$) or decay ($r < 1$).





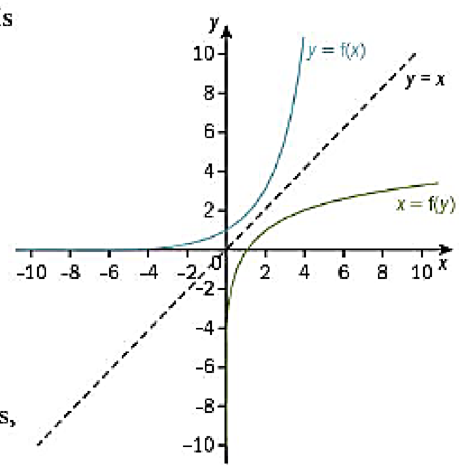
$f(t) = 1.4 \times 1.3^t$ The initial value of the function, a, is 1.4. We also see that $r = 1.3 > 1$, meaning this function describes exponential growth.

$f(t) = 5 \times 0.8^t$ The initial value of the function, a, is now 5 and $r = 0.8 < 1$, meaning this function describes exponential decay.

The Logarithmic Function

The graph shows the function $y = 2^x$ in blue, and the inverse of this function in green (recall that the inverse function is the mirror image of the function in the line $y = x$).

This inverse function is known as the **logarithmic function**.



The logarithmic function is the inverse of the exponential function to the same base.

The green graph shown here is the logarithm to base 2, because the blue graph shows 2^x.

The exponential function has domain the set of all real numbers, with range the positive real numbers, so it follows that the logarithmic function is defined on a domain of the positive real numbers and its range is the set of all real numbers.

Formally, we define the logarithmic function by

$$y = b^x \Leftrightarrow x = \log_b y, \text{ where } x \in \mathbb{R},\ y \in \mathbb{R},\ y > 0$$

$\log_a a = 1$	$\log_a 1 = 0$
$\log_a a^x = x$	$a^{\log_a x} = x$

Example 8

Convert $5^3 = 125$ to logarithmic form.

Example 9

Write in logarithmic form: (a) $3^2 = 9$, *(b)* $x^3 = 10$. *(c)* $2^{-2} = \frac{1}{4}$.

Example 10

Convert $\log_3 x = 2.5$ to exponential form.

Example 11

Write in exponential form: (a) $4 = \log_3 x$, *(b)* $x = \log_5 7$, *(c)* $2 = \log_x 5$.

Example 12

a) Write these in the form $y = b^x$.

i) $\log_2 64 = 6$ **ii)** $\log_k m = p$

b) Write these in the form $x = \log_b y$.

i) $5^{-3} = 0.008$ **ii)** $h^5 = R$

Example 12

Find the value of

a) $\log_2 16$ **b)** $\log_3 243$ **c)** $\log_{10} 1000$ **d)** $\log_7 343$.

Example 13

Find the value of:

a $\log_2 16$

b $\log_3 \frac{1}{9}$

Example 14

Simplify $\log_x\left(\frac{\sqrt[3]{x}}{x}\right)$.

Laws of Logarithms

Multiplication law

$\log_a(xy) = \log_a x + \log_a y$

Division law

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

Power law

$\log_a(x)^m = m\log_a x$

$\log_a\left(\frac{1}{x}\right) = -\log_a x$

Example 15

Given that $\log_4 p = x$ and $\log_4 q = y$, express in terms of x and/or y

a $\log_4 p^5 - \log_4 q^2$ **b** $\log_4 \sqrt{p} + 5\log_4 \sqrt[3]{q}$ **c** $\log_4\left(\frac{64}{p}\right)$.

Example 16

Use the laws of logarithms to simplify these expressions.

a $\log_2 3 + \log_2 5$ **b** $\log_3 8 - \log_3 4$ **c** $2\log_5 2 + \log_5 3$

Example 17

Example 9

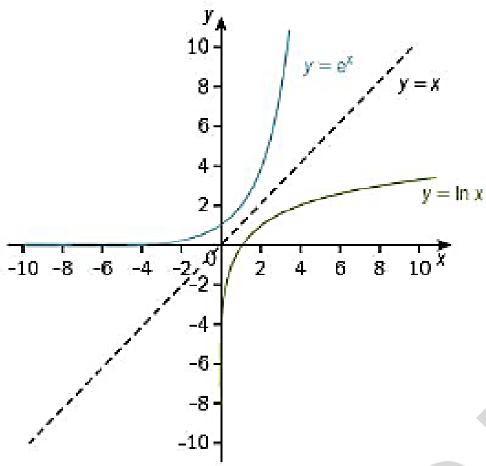
Express these in terms of $\log x$, $\log y$, and $\log z$.

a) $\log xy$ **b)** $\log\left(\frac{xy^2}{z}\right)$ **c)** $\log\sqrt{\left(\frac{x^3y^2}{z^4}\right)}$

2.3 e^x and logarithms to base e

You have seen that you can use any positive real number as a base for a power or logarithmic function, but in practice two bases are very commonly used, and they are available on your calculator.

- The button marked $\boxed{\text{log}}$ uses base 10 since our number system is based on powers of 10.
- The $\boxed{\text{ln}}$ button uses base e (= 2.71828 ...). These are sometimes called natural logarithms because of some special properties the number represented by 'e' has when used in an exponential function.



The blue graph is $y = e^x$ and the green graph is $y = \log_e x$ (also known as $y = \ln x$). These are inverses and are similar to the graphs you saw earlier at the beginning of section 2.2. One feature to notice about $y = e^x$ is that as x increases, not only does e^x also increase, but its rate of increase (the gradient of the graph) gets larger.

$\ln x$ or $\log_e x$ is the inverse function of e^x.

KEY POINT 2.5

If $y = e^x$ then $x = \ln y$

Solving Equations using Logarithms

Example 18

Solve the equation $4^x = 16$ by taking logarithms of both sides.

Example 19

Solve the equation $5^x = 3^{2x+1}$.

Example 20

Solve, giving your answers correct to 3 significant figures.

a $5^{2x+1} = 7$

b $3^{2x} = 4^{x+5}$

Example 21

Solve the equation $2(3^{2x}) + 7(3^x) = 15$, giving your answers correct to 3 significant figures.

Example 22

Solve:

a $2\log_8(x+2) = \log_8(2x+19)$

b $4\log_x 2 - \log_x 4 = 2$

Example 23

Solve the equation $\log_{10}(2+x) = 2 + \log_{10} x$.

Example 24

Solve the equation $\log_a(2-x) - 2\log_a x = \log_a 3$.

Example 25

Solve the equation $\log_{10}(2-x) - 2\log_{10} x = 1$.

Solving inequalities using logarithms

Example 26

Solve the inequality $0.6^x < 0.7$, giving your answer in terms of base 10 logarithms.

Example 27

Solve the inequality $4 \times 3^{2x-1} > 5$, giving your answer in terms of base 10 logarithms.

Example 28

Solve the following inequalities.

a) $5^x \leq 13.3$ **b)** $(0.4)^x < 0.0001$

Applications of Logarithms

Example 29

A sum of money \$P is invested at a compound interest of r% per year.

(a) Show that it will amount to $\$P(1 + \frac{r}{100})^n$ *after n years.*

(b) If the rate of interest is 8%, after how many years will the sum of money be doubled?

Example 30

Given that $y = ax^b + 3$ *where* $a > 0$ *and that* $y = 8$ *when* $x = 2$ *and* $y = 48$ *when* $x = 8$, *find the values of a and b.*

Example 31

4 Use logarithms to solve the equation

$$5^{x+3} = 7^{x-1},$$

giving the answer correct to 3 significant figures. [4]

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q1 November 2015

Example 32

8 i Given that $y = 2^x$, show that the equation

$$2^x + 3(2^{-x}) = 4$$

can be written in the form

$$y^2 - 4y + 3 = 0.$$ [3]

ii Hence solve the equation

$$2^x + 3(2^{-x}) = 4,$$

giving the values of x correct to 3 significant figures where appropriate. [3]

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Example 33

November 2014/31 Question 1 **2**

1 Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. [3]

Example 34

June 2014/31 Question 6

6 It is given that $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$.

(i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. [3]

(ii) By f rst using the factor theorem, factorise $16x^3 - 24x^2 - 15x - 2$ completely. [4]

(iii) Hence solve the equation $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$. [1]

Example 35

June 2014/32 Question 2

2 Solve the equation

$$2\ln(5 - e^{-2x}) = 1,$$

giving your answer correct to 3 signif cant f gures. [4]

Example 36

June 2014/33 Question 1

1 Solve the equation $\log_{10}(x + 9) = 2 + \log_{10} x$. [3]

CHAPTER 6: LINEAR LAW

Using Logarithms to reduce equations to linear form

Many scientific, economic, and social science quantities can be described (at least approximately) by relationships which follow either an exponential growth or decay law, or else a power law. If we take logarithms of an equation which follows either of these two laws, we transform it into a linear expression.
This allows values of unknown constants to be estimated from observational data.

Exponential growth or decay:

$y = ab^t \Rightarrow \log y = \log a + t \log b$

The graph of '$\log y$' against 't' has intercept '$\log a$' and gradient '$\log b$'.

Power law:

$y = ax^n \Rightarrow \log y = \log a + n \log x$

The graph is of '$\log y$' against '$\log x$' and has intercept '$\log a$' and gradient n.

You know some examples of power laws in geometrical formulae for areas and volumes.

Example 1

By taking logarithms, transform these relationships between the two stated variables into a linear relationship between two new variables, and state the new variables.

a) y and t are related by $y = 7b^t$.

b) y and x are related by $y = ax^3$.

Example 2

In order that each of the equations

(i) $y = ab^x$,

(ii) $y = Ax^k$,

(iii) $px + qy = xy$,

where a, b, A, k, p and q are unknown constants, may be represented by a straight line, they each need to be expressed in the form $Y = mX + c$, where X and Y are each functions of x and/or y, and m and c are constants. Copy the following table and insert in it an expression for Y, X, m and c for each case.

	Y	X	m	c
$y = ab^x$				
$y = Ax^k$				
$px + qy = xy$				

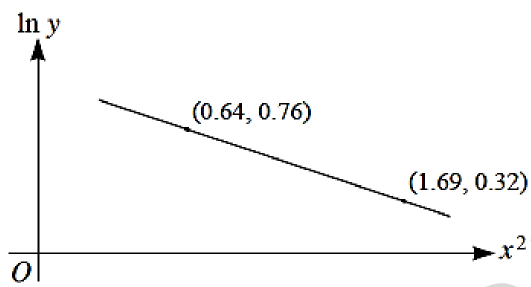
[7]

Example 3

For the following linear equations involving logarithms, find the relationship between the unknown variables, giving your answer in a form not involving logarithms.

a) $\log_{10} y = 2 + 3\log_{10} x$ **b)** $\log_{10} A = \log_{10} \pi + 2\log_{10} r$ **c)** $\log_{10} y = 0.3 + 0.7x$

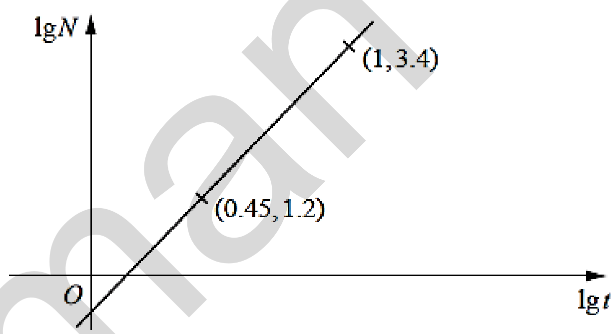
Example 4



The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points (0.64, 0.76) and (1.69, 0.32), as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]

Example 5

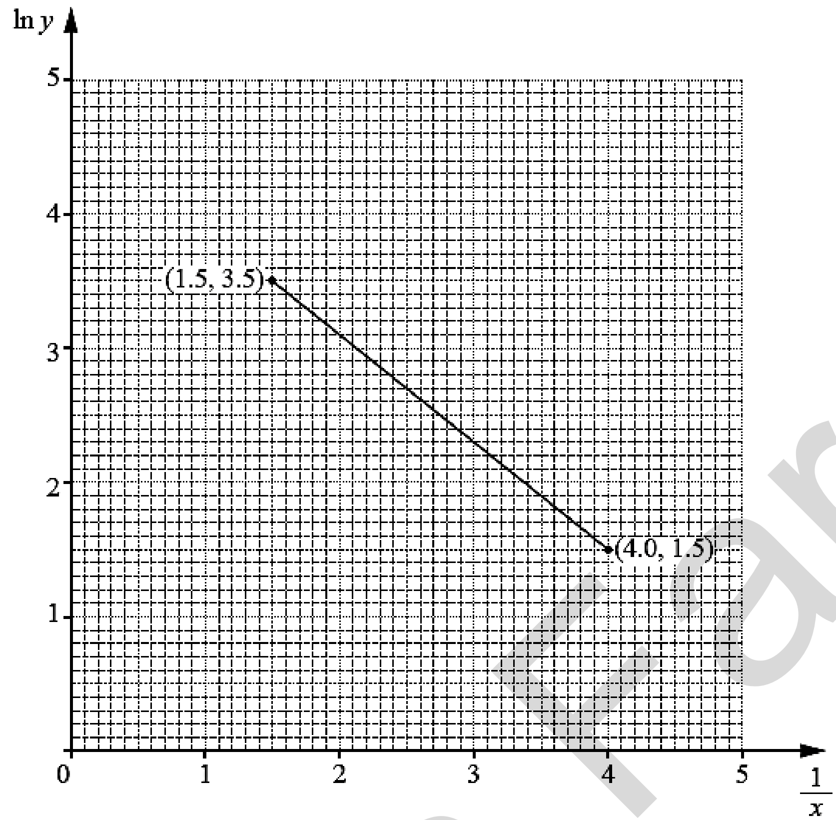
Variables t and N are such that when $\lg N$ is plotted against $\lg t$, a straight line graph passing through the points (0.45, 1.2) and (1, 3.4) is obtained.



(i) Express the equation of the straight line graph in the form $\lg N = m \lg t + \lg c$, where m and c are constants to be found. [4]

(ii) Hence express N in terms of t. [1]

Example 6



The variables x and y are such that when $\ln y$ is plotted against $\frac{1}{x}$ the straight line graph shown above is obtained.

(i) Given that $y = Ae^{\frac{b}{x}}$, find the value of A and of b. [4]

Example 7

Fig.16.7 shows the straight line obtained by plotting lg y against lg x. Find

(a) lg y in terms of lg x,

(b) y in terms of x,

(c) the value of x when y = 700.

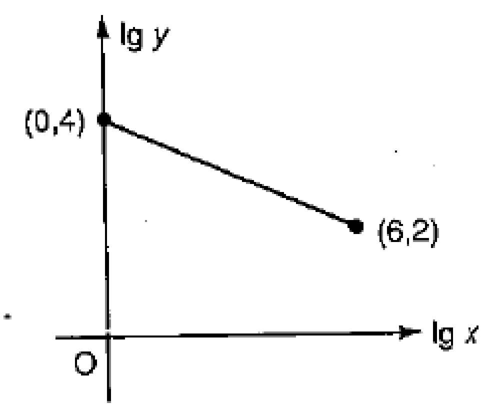


Fig. 16.7

Example 8

The variables x and y satisfy the relation $5^y = 3^{2x+1}$. By taking logarithms, show that the graph of y against x is a straight line, and find the exact values of the gradient and the intercept.

Example 9

When $\ln y$ is plotted against x^2 a straight line is obtained which passes through the points (0.2, 2.4) and (0.8, 0.9).

(i) Express $\ln y$ in the form $px^2 + q$, where p and q are constants. [3]

(ii) Hence express y in terms of z, where $z = e^{x^2}$. [3]

HOMEWORK: EXPONENTS AND LOGARITHMS

VARIANT 32

1 **(i)** Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in y. [2]

(ii) Hence solve the equation

$$2^x - 2^{-x} = 1.$$ [4]

Answers: **(i)** $y^2 - y - 1 = 0$; **(ii)** 0.694. J04/Q4

2 Solve the equation

$$\ln(1 + x) = 1 + \ln x,$$

giving your answer correct to 2 significant figures. [4]

Answer: 0.58. N04/Q2

3 Given that $x = 4(3^{-y})$, express y in terms of x. [3]

Answer: $\dfrac{\ln 4 - \ln x}{\ln 3}$. J06/Q1

4 Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation

$$3^x = 2 + 3^{-x}.$$ [6]

Answer: 0.802. J07/Q4

5 Solve, correct to 3 significant figures, the equation

$$e^x + e^{2x} = e^{3x}.$$ [5]

Answer: 0.481. J08/Q2

6 Solve the equation

$$\ln(x + 2) = 2 + \ln x,$$

giving your answer correct to 3 decimal places. [3]

Answer: 0.313 . N08/Q1

7 Solve the equation $\ln(2 + e^{-x}) = 2$, giving your answer correct to 2 decimal places. [4]

Answer: -1.68. J09/Q1

8 Solve the equation

$$\ln(5 - x) = \ln 5 - \ln x,$$

giving your answers correct to 3 significant figures. [4]

Answers: 1.38, 3.62. N09/32/Q1

9 Solve the equation

$$\ln(1 + x^2) = 1 + 2\ln x,$$

giving your answer correct to 3 significant figures. [4]

Answer: 0.763. N10/32/Q2

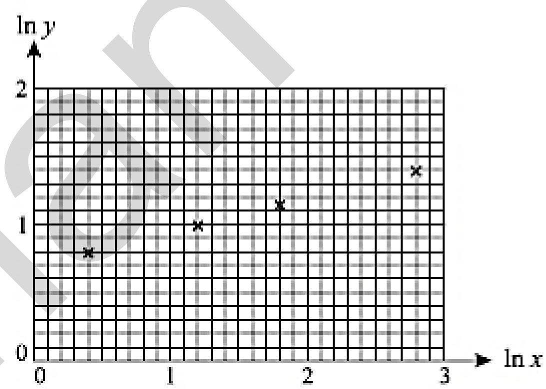
10 Solve the equation

$$\frac{2^x + 1}{2^x - 1} = 5,$$

giving your answer correct to 3 significant figures. [4]

Answer: 0.585 J10/32/Q1

11



Two variable quantities x and y are related by the equation $y = Ax^n$, where A and n are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of x and y. Use the diagram to estimate the values of A and n. [5]

Answers: $A = 2.01$; $n = 0.25$. N05/Q2

12 Solve the inequality $|2^x - 8| < 5$. [4]

Answer: $1.58 < x < 3.70$. N03/Q1

13 (i) Show that the equation

$$\log_{10}(x+5) = 2 - \log_{10} x$$

may be written as a quadratic equation in x. [3]

(ii) Hence find the value of x satisfying the equation

$$\log_{10}(x+5) = 2 - \log_{10} x.$$ [2]

Answers: (i) $x^2+5x-100=0$; (ii) 7.81 . N02/Q3

14 Find the set of values of x satisfying the inequality $|3^x - 8| < 0.5$, giving 3 significant figures in your answer. [4]

Answer: $1.83 < x < 1.95$. N06/Q1

15 (i) Show that the equation

$$\log_2(x+5) = 5 - \log_2 x$$

can be written as a quadratic equation in x. [3]

(ii) Hence solve the equation

$$\log_2(x+5) = 5 - \log_2 x.$$ [2]

Answer: **(ii)** 3.68. J11/32/Q2

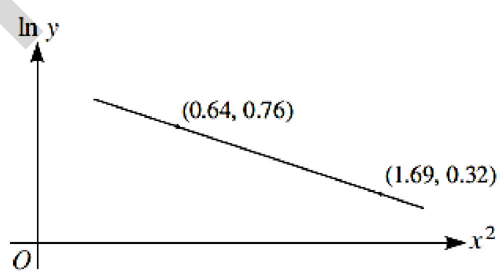
16 Solve the equation

$$\ln(3x+4) = 2\ln(x+1),$$

giving your answer correct to 3 significant figures. [4]

Answer: 2.30 J12/32/Q1

17



The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points (0.64, 0.76) and (1.69, 0.32), as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]

Answer: $A = 2.80$, $k = 0.42$ J13/32/Q3

18 Solve the equation

$$2\ln(5 - e^{-2x}) = 1,$$

giving your answer correct to 3 significant figures. [4]

Answer: –0.605 **J14/32/Q2**

19 Using the substitution $u = 4^x$, solve the equation $4^x + 4^2 = 4^{x+2}$, giving your answer correct to 3 significant figures. [4]

Answer: 0.0466 **J15/32/Q2**

20 Using the substitution $u = e^x$, or otherwise, solve the equation

$$e^x = 1 + 6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

Answer: 1.10 **N11/32/Q1**

21 Solve the equation

$$5^{x-1} = 5^x - 5,$$

giving your answer correct to 3 significant figures. [4]

Answer: 1.14. **N12/32/Q2**

22 Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

Answer: 0.631, -0.369 **N13/32/Q2**

23 Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. [3]

Answer: 22.281 **N14/32/Q1**

24 Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. [5]

Answer: 0.438 **N15/32/Q2**

HOMEWORK: EXPONENTS AND LOGARITHMS– VARIANTS 31 & 33

1 Solve the equation $\ln(x+4) = 2\ln x + \ln 4$, giving your answer correct to 3 significant figures. [4]

Answer: $x = 1.13$ 33/J15/1

2 Use logarithms to solve the equation $2^{5x} = 3^{2x+1}$, giving the answer correct to 3 significant figures. [4]

Answer: 0.866 31/J15/1

3 (i) Sketch the curve $y = \ln(x+1)$ and hence, by sketching a second curve, show that the equation

$$x^3 + \ln(x+1) = 40$$

has exactly one real root. State the equation of the second curve. [3]

Answers: (i) $y = 40 - x^3$, (iii) 3.377, (iv) 1.48 33/N14/9

4 Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. [3]

Answer: 22.281 31/N14/1

5 Solve the equation $\log_{10}(x+9) = 2 + \log_{10} x$. [3]

Answer: $x = \frac{1}{11}$ 33/J14/1

6 It is given that $2\ln(4x-5) + \ln(x+1) = 3\ln 3$.

(i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. [3]

Answer: 31/J14/6

7 Given that $2\ln(x+4) - \ln x = \ln(x+a)$, express x in terms of a. [4]

Answer: $\frac{16}{a-8}$ 33/N13/1

8 It is given that $\ln(y+1) - \ln y = 1 + 3\ln x$. Express y in terms of x, in a form not involving logarithms. [4]

Answer: $y = (ex^3 - 1)^{-1}$ 33/J13/2

9 (i) Solve the equation $|4x-1|=|x-3|$. [3]

(ii) Hence solve the equation $|4^{y+1}-1|=|4^y-3|$ correct to 3 significant figures. [3]

Answer: $-\frac{2}{3}$ and $\frac{4}{5}$ *Answer*: -0.161 31/J13/4

10 Solve the equation

$$\ln(x+5)=1+\ln x,$$

giving your answer in terms of e. [3]

Answer: $\frac{5}{e-1}$. 33/N12/1

11 Solve the equation

$$5^{x-1}=5^x-5,$$

giving your answer correct to 3 significant figures. [4]

Answer: 1.14. 31/N12/Q2

12 Solve the equation $\ln(2x+3)=2\ln x+\ln 3$, giving your answer correct to 3 significant figures. [4]

Answer: 1.39 33/J12/2

13 Solve the equation $|4-2^x|=10$, giving your answer correct to 3 significant figures. [3]

Answer: 3.81 31/J12/1

14 Using the substitution $u=e^x$, or otherwise, solve the equation

$$e^x=1+6e^{-x},$$

giving your answer correct to 3 significant figures. [4]

Answer: 1.10 31/N11/1

15 Use logarithms to solve the equation $5^{2x-1}=2(3^x)$, giving your answer correct to 3 significant figures. [4]

Answer: 1.09. 33/J11/1

16 The curve with equation

$$6e^{2x}+ke^y+e^{2y}=c,$$

where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

(i) Show that $58+2k=c$. [2]

31/J11/Q5

17 Solve the equation

$$\ln(1 + x^2) = 1 + 2\ln x,$$

giving your answer correct to 3 significant figures. [4]

Answer: 0.763.

31/N10/2

18 The variables x and y satisfy the equation $y^3 = Ae^{2x}$, where A is a constant. The graph of $\ln y$ against x is a straight line.

(i) Find the gradient of this line. [2]

(ii) Given that the line intersects the axis of $\ln y$ at the point where $\ln y = 0.5$, find the value of A correct to 2 decimal places. [2]

Answers: **(i)** $\frac{2}{3}$, **(ii)** 4.48.

33/J10/2

19 The variables x and y satisfy the equation $x^n y = C$, where n and C are constants. When $x = 1.10$, $y = 5.20$, and when $x = 3.20$, $y = 1.05$.

(i) Find the values of n and C. [5]

(ii) Explain why the graph of $\ln y$ against $\ln x$ is a straight line. [1]

Answers: **(i)** 1.50, 6.00.

31/J10/3

20 Sketch the graph of $y = e^{ax} - 1$ where a is a positive constant. [2]

N15/33/Q1

21 It is given that $z = \ln(y + 2) - \ln(y + 1)$. Express y in terms of z. [3]

Answer: $y = \frac{2 - e^z}{e^z - 1}$

N16/33/Q1

22 The variables x and y satisfy the relation $3^y = 4^{2-x}$.

(i) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

(ii) Calculate the exact x-coordinate of the point of intersection of this line with the line with equation $y = 2x$, simplifying your answer. [2]

Answer: **(i)** $y\ln 3 = 2\ln 4 - x\ln 4$, $-\frac{\ln 4}{\ln 3}$; **(ii)** $x = \frac{\ln 4}{\ln 6}$ or $\log_{36}16$, etc.

J16/33/Q2

23 Using the substitution $u = e^x$, solve the equation $4e^{-x} = 3e^x + 4$. Give your answer correct to 3 significant figures. [4]

Answer: $x = -0.405$ **J17/33/Q3**

24 Showing all necessary working, solve the equation $\ln(x^4 - 4) = 4\ln x - \ln 4$, giving your answer correct to 2 decimal places. [4]

Answer: 1.52 **J18/31/Q1**

25 Showing all necessary working, solve the equation $5^{2x} = 5^x + 5$. Give your answer correct to 3 decimal places. [5]

Answer: $x = 0.638$ **J18/33/Q2**

v

CHAPTER 7: TRIGONOMETRY

Example 1

Solve

a) $4\sin\theta = 2\cos\theta$ for $0° \le \theta < 360°$

b) $2(\sin^2\theta - \cos^2\theta) = 1$ for $0 \le \theta \le 2\pi$.

Secant, Cosecant & Cotangent

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \left(\frac{\cos\theta}{\sin\theta}\right)$$

Example 2

Solve for $0° \le \theta \le 360°$.

a) $\sec\theta = 2$ **b)** $\cot^2\theta = 3$ **c)** $11 + 3\operatorname{cosec}2\theta = 1$

Further Trigonometric Identities

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$$

Example 3

Prove the identity $(\tan\theta + \cot\theta)^2 \equiv \sec^2\theta + \operatorname{cosec}^2\theta$.

Example 4

Solve the equation $\sec^2\theta + \tan\theta - 1 = 0$ for $0° \le \theta \le 360°$.

Example 5

Solve $\sec^2 x - \tan x - 3 = 0$ for $0° \leqslant x \leqslant 360°$.

Compound angle formulae

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Example 6

Prove that $2\cos\left(\theta - \frac{\pi}{3}\right) \equiv \cos\theta + \sqrt{3}\sin\theta$.

Example 7

Given that $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, where A is obtuse and B is acute, find the value of:

a $\sin(A+B)$ **b** $\cos(A-B)$ **c** $\tan(A-B)$

Example 8

By considering $\sin(45° + 30°)$, prove that $\sin 75° = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

Example 9

Solve the equation $\cos(\theta + 60°) = 2\sin(\theta - 45°)$ for $0° \le \theta \le 360°$.

Example 10

Solve the equation $\sin(60° - x) = 2\sin x$ for $0° < x < 360°$.

Double angle formulae

$\sin 2A \equiv 2\sin A\cos A$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Example 11

Prove that **a)** $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

Example 12

. Prove that $\sec 2\theta - \tan 2\theta = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$.

Example 13

Given that $\sin x = -\dfrac{3}{5}$ and that $180° < x < 270°$, find the exact value of:

a $\sin 2x$ **b** $\cos 2x$ **c** $\tan\dfrac{x}{2}$

Example 14

Solve the following equations for $0° \le \theta \le 360°$.

a) $4\sin 2\theta = \sin\theta$ **b)** $\cos 2\theta = \cos\theta$

Example 15

Solve the equation $\sin 2x = \sin x$ for $0° \leqslant x \leqslant 360°$.

Example 16

Solve the equation $\cos 2x + 3\sin x = 2$ for $0 \leqslant x \leqslant 2\pi$.

Example 17

Express $\cos 3\theta$ in terms of $\cos\theta$.

Example 18

Prove the identity $\tan 3x \equiv \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

Example 19

Prove the identity $\dfrac{2\sin(x-y)}{\cos(x-y)-\cos(x+y)} \equiv \cot y - \cot x$.

Example 20

8 i Show that the equation $\tan(x+45°) = 6\tan x$ can be written in the form $6\tan^2 x - 5\tan x + 1 = 0$. [3]

ii Hence solve the equation $\tan(x+45°) = 6\tan x$, for $0° < x < 180°$. [3]

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q3 June 2010

Example 21

By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

Cambridge International A Level Mathematics 9709 Paper 31 Q3 June 2016

Function of Trigonometry

$$a\sin\theta \pm b\cos\theta \equiv R\sin(\theta \pm \alpha) \text{ and } a\cos\theta \pm b\sin\theta \equiv R\cos(\theta \mp \alpha)$$

where $R = \sqrt{a^2+b^2}$, $\tan\alpha = \frac{b}{a}$ and $0° < \alpha < 90°$.

Example 22

a) Express $2\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and $0° \le \alpha \le 90°$, giving the exact value of R and the value of α correct to 2 decimal places.

b) Hence solve the equation $2\sin\theta + \cos\theta = 2$ for $0° \le \theta \le 360°$.

Example 23

a Express $2\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places.

b Hence solve the equation $2\cos\theta - 3\sin\theta = 1.3$ for $0° < \theta < 360°$.

Example 24

10 i Express $3\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

ii Hence solve the equation $3\cos 2x + \sin 2x = 2$, giving all solutions in the interval $0° \leqslant x \leqslant 360°$. [5]

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q7 November 2013

Example 25

June 2013/32 Question 7

7 **(i)** By fi st expanding $\cos(x + 45°)$, express $\cos(x + 45°) - (\sqrt{2})\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 signifi ant figu es and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$\cos(x + 45°) - (\sqrt{2})\sin x = 2,$$

for $0° < x < 360°$. [4]

Example 26

a) Express $3\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $R > 0$ and $0° \le \alpha \le 90°$, stating the exact values of R and α.

b) Determine the greatest and least possible values of $[(3\cos\theta + \sqrt{3}\sin\theta)^2 - 5]$ as θ varies.

The maximum value of $a\sin\theta + b\cos\theta$ is $\sqrt{a^2+b^2}$ and occurs when $\sin(\theta+\alpha) = 1$.

The minimum value of $a\sin\theta + b\cos\theta$ is $-\sqrt{a^2+b^2}$ and occurs when $\sin(\theta+\alpha) = -1$.

Example 27

Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

Hence, find the maximum and minimum values of the expression $2\cos\theta - \sin\theta$ and the values of θ in the interval $0° < \theta < 360°$ for which these occur.

Example 28

13 i Express $4\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.
Give the exact value of R and the value of α correct to 2 decimal places. [3]

ii Solve the equation $4\sin\theta - 6\cos\theta = 3$ for $0° \leqslant \theta \leqslant 360°$. [4]

iii Find the greatest and least possible values of $(4\sin\theta - 6\cos\theta)^2 + 8$ as θ varies. [2]

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q8 June 2011

Example 29

14 i By first expanding $\sin(2\theta + \theta)$, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [4]

ii Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

iii Hence solve the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$, giving your answers correct to 3 significant figures. [4]

Cambridge International A Level Mathematics 9709 Paper 31 Q8 November 2014

Example 30

November 2014/33 Question 4

4 **(i)** Show that $\cos(\theta - 60°) + \cos(\theta + 60°) \equiv \cos\theta$. [3]

(ii) Given that $\dfrac{\cos(2x - 60°) + \cos(2x + 60°)}{\cos(x - 60°) + \cos(x + 60°)} = 3$, find the exact value of $\cos x$. [4]

Example 31

June 2014/31 Question 1

1 (i) Simplify $\sin 2\alpha \sec \alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos \beta = 0$, f nd the exact value of $\cos \beta$. [3]

Example 32

June 2014/32 Question 3

3 Solve the equation

$$\cos(x + 30°) = 2\cos x,$$

giving all solutions in the interval $-180° < x < 180°$. [5]

Example 33

June 2014/31 Question 8

8 (i) By sketching each of the graphs $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval $0 < x < \pi$. [3]

HOMEWORK: TRIGONOMETRY VARIANT 32

1 (i) Show that the equation

$$\sin(x - 60^\circ) - \cos(30^\circ - x) = 1$$

can be written in the form $\cos x = k$, where k is a constant. [2]

(ii) Hence solve the equation, for $0^\circ < x < 180^\circ$. [2]

Answer: **(ii)** 125.3°. **J03/Q1**

2 (i) Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1 - \cos 4\theta).$$ [3]

J04/Q5

3 (i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3.$$ [4]

(ii) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 2,$$

for $0^\circ \leqslant \theta \leqslant 360^\circ$. [4]

Answers: **(ii)** 27.2°, 152.8°, 207.2°, 332.8°. **J05/Q6**

4 (i) Prove the identity $\operatorname{cosec} 2\theta + \cot 2\theta \equiv \cot\theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec} 2\theta + \cot 2\theta = 2$, for $0^\circ \leqslant \theta \leqslant 360^\circ$. [2]

Answer: (ii) 26.6°, 206.6°. **J09/Q3**

5 Solve the equation

$$\cos\theta + 3\cos 2\theta = 2,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 180^\circ$. [5]

Answers: 33.6°, 180°. **N03/Q3**

6 (i) Show that the equation

$$\tan(45^\circ + x) = 2\tan(45^\circ - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0.$$ [4]

(ii) Hence solve the equation $\tan(45^\circ + x) = 2\tan(45^\circ - x)$, for $0^\circ < x < 90^\circ$. [3]

Answer: (ii) 9.7°, 80.3°. N04/Q4

7 Sketch the graph of $y = \sec x$, for $0 \leqslant x \leqslant 2\pi$. [3]

J04/Q1

8 By expressing $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, solve the equation

$$8\sin\theta - 6\cos\theta = 7,$$

for $0^\circ \leqslant \theta \leqslant 360^\circ$. [7]

Answer: θ = 81.3° or 172.4°. N05/Q5

9 (i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$. [4]

Answers: (i) $R = 25$, $\alpha = 73.74°$; (ii) 20.6°, 126.9°. J06/Q4

10 Solve the equation

$$\tan x \tan 2x = 1,$$

giving all solutions in the interval $0^\circ < x < 180^\circ$. [4]

Answers: 30°, 150°. N06/Q2

11 (i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

Answer: (i) $2\cos\left(\theta - \frac{1}{3}\pi\right)$. J07/Q5

12 (i) Show that the equation

$$\tan(45^\circ + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(45^\circ + x) - \tan x = 2,$$

giving all solutions in the interval $0^\circ \leqslant x \leqslant 180^\circ$. [4]

Answers: (ii) 22.5°, 112.5°.	N07/Q5

13 (i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form

$$\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0. \qquad [4]$$

(ii) Hence, or otherwise, solve the equation

$$\tan(30° + \theta) = 2\tan(60° - \theta),$$

for $0° \leqslant \theta \leqslant 180°$. [3]

Answer: (ii) 24.7°, 95.3°.	J08/Q4

14 (i) Express $5\sin x + 12\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Answers: (i) $13\sin(x + 67.38°)$; (ii) 27.4°, 175.2°.	N08/Q6

15 The angles α and β lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha + \beta) = 3.$$

Find the possible values of α and β. [6]

Answers: $\alpha = 45°$, $\beta = 26.6°$ and $\alpha = 116.6°$, $\beta = 135°$.	N09/Q4

16 It is given that $\cos a = \frac{3}{5}$, where $0° < a < 90°$. Showing your working and without using a calculator to evaluate a,

(i) find the exact value of $\sin(a - 30°)$, [3]

(ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

Answers: (i) $\frac{1}{10}(4\sqrt{3} - 3)$; (ii) $-\frac{24}{7}$, $-\frac{44}{117}$	J10/32/Q3

17 Solve the equation

$$\cos(\theta + 60°) = 2\sin\theta,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 360°$. [5]

Answer: 9.9°, 189.9°.	N10/32/Q3

18 **(i)** Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the value of α correct to 2 decimal places. [3]

Hence

(ii) solve the equation

$$4\sin\theta - 3\cos\theta = 2,$$

giving all values of θ such that $0° < \theta < 360°$, [4]

(iii) write down the greatest value of $\dfrac{1}{4\sin\theta - 3\cos\theta + 6}$. [1]

Answers: (i) $5\sin(\theta - 36.87°)$; (ii) 60.4° and 193.3°; (iii) 1. N02/Q5

19 Solve the equation

$$\cos\theta + 4\cos 2\theta = 3,$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [5]

Answer: 29.0°, 180°. J11/32/Q3

20 Solve the equation

$$\operatorname{cosec} 2\theta = \sec\theta + \cot\theta,$$

giving all solutions in the interval $0° < \theta < 360°$. [6]

Answers: 201.5°, 338.5° J12/32/Q4

21 **(i)** By first expanding $\cos(x + 45°)$, express $\cos(x + 45°) - (\sqrt{2})\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 significant figures and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$\cos(x + 45°) - (\sqrt{2})\sin x = 2,$$

for $0° < x < 360°$. [4]

Answers: **(i)** $R = 2.236$, $\alpha = 71.57°$; **(ii)** 261.9°. 315° J13/32/Q7

22 Solve the equation

$$\cos(x + 30°) = 2\cos x,$$

giving all solutions in the interval $-180° < x < 180°$. [5]

Answer: −66.2°, 113.8° J14/32/Q3

23 **(i)** Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, stating the exact value of R and giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$3\sin\theta + 2\cos\theta = 1,$$

for $0° < \theta < 180°$. [3]

Answer: **(i)** 33.69° **(ii)** 130.2° **J15/32/Q4**

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24 **(i)** Express $\cos x + 3\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3\sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

Answer: **(i)** $R = \sqrt{10}$, $\alpha = 71.57°$; **(ii)** 10.4°, 61.2° **N11/32/Q6**

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25 Solve the equation

$$\sin(\theta + 45°) = 2\cos(\theta - 30°),$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Answer: 105.9°. **N12/32/Q3**

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26 **(i)** By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \tfrac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

Answers: **(iii)** 0.322, 0.799, – 1.12 **N14/32/Q8**

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27 The angles θ and ϕ lie between 0° and 180°, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of θ and ϕ. [6]

Answer: $\theta = 53.1°, \phi = 161.6°$; $\theta = 135°, \phi = 63.4°$ **N15/32/Q3**

28 The angles A and B are such that

$$\sin(A + 45°) = (2\sqrt{2})\cos A \quad \text{and} \quad 4\sec^2 B + 5 = 12\tan B.$$

Without using a calculator, find the exact value of $\tan(A - B)$. [8]

Answer: $\frac{3}{11}$ N15/33/Q6

TRIGONOMETRY– VARIANTS 31 & 33

1 Solve the equation $\cot 2x + \cot x = 3$ for $0° < x < 180°$.

Answers: 24.9° and 98.8° 33/J15/3

2 **(i)** Show that $\cos(\theta - 60°) + \cos(\theta + 60°) \equiv \cos\theta$. [3]

(ii) Given that $\dfrac{\cos(2x - 60°) + \cos(2x + 60°)}{\cos(x - 60°) + \cos(x + 60°)} = 3$, find the exact value of $\cos x$. [4]

Answer: (ii) $\frac{1}{4}\left(3 - \sqrt{17}\right)$ 33/N14/4

3 **(i)** By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \tfrac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

Answers: (iii) 0.322, 0.799, – 1.12 31/N14/8

4 **(i)** Show that the equation

$$\tan(x - 60°) + \cot x = \sqrt{3}$$

can be written in the form

$$2\tan^2 x + (\sqrt{3})\tan x - 1 = 0.$$ [3]

(ii) Hence solve the equation

$$\tan(x - 60°) + \cot x = \sqrt{3},$$

for $0° < x < 180°$. [3]

Answer: 21.6° and 128.4° 33/J14/3

5 **(i)** Simplify $\sin 2\alpha \sec\alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos\beta = 0$, find the exact value of $\cos\beta$. [3]

Answer: (i) 2sinα (ii) $\frac{1}{3}$ 31/J14/1

6 (i) Given that $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$, show that $2\sin\theta + 4\cos\theta = 3$. [3]

(ii) Express $2\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$ where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(iii) Hence solve the equation $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$ for $0° < \theta < 360°$. [4]

Answer: (ii) $R = \sqrt{20}$, $\alpha = 63.43°$ (iii) 74.4°, 338.7° 33/N13/7

7 Solve the equation $\tan 2x = 5\cot x$, for $0° < x < 180°$. [5]

Answer: 40.2° and 139.8° 33/J13/3

8 (i) Express $(\sqrt{3})\cos x + \sin x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

Answer: 2 and $\frac{\pi}{6}$ 33/J13/4

9 (i) Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. [3]

(ii) Hence

Answer: 0.6435 *Answer*: 1.80 and 5.77 31/J13/9

10 (i) Express $24\sin\theta - 7\cos\theta$ in the form $R\sin(\theta-\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence find the smallest positive value of θ satisfying the equation

$$24\sin\theta - 7\cos\theta = 17.$$ [2]

Answers: (i) $25\sin(\theta - 16.26°)$; (ii) 59.1°. 33/N12/2

11 Solve the equation

$$\sin(\theta + 45°) = 2\cos(\theta - 30°),$$

giving all solutions in the interval $0° < \theta < 180°$. [5]

Answer: 105.9°. 31/N12/3

12 It is given that $\tan 3x = k \tan x$, where k is a constant and $\tan x \neq 0$.

(i) By first expanding $\tan(2x + x)$, show that

$$(3k - 1)\tan^2 x = k - 3.$$ [4]

(ii) Hence solve the equation $\tan 3x = k \tan x$ when $k = 4$, giving all solutions in the interval $0° < x < 180°$. [3]

(iii) Show that the equation $\tan 3x = k \tan x$ has no root in the interval $0° < x < 180°$ when $k = 2$. [1]

Answers: **(ii)** 16.8°, 163.2° 33/J12/6

13 (i) Express $8 \cos \theta + 15 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $8 \cos \theta + 15 \sin \theta = 12$, giving all solutions in the interval $0° < \theta < 360°$. [4]

Answer: **(i)** $17\cos(\theta - 61.93°)$; **(ii)** 107.0°, 16.8° 33/N11/3

14 (i) Express $\cos x + 3 \sin x$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $\cos 2\theta + 3 \sin 2\theta = 2$, for $0° < \theta < 90°$. [5]

Answer: (i) $R = \sqrt{10}$, $\alpha = 71.57°$; **(ii)** 10.4°, 61.2° 31/N11/6

15 (i) Show that the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = k$$

can be written in the form

$$(2\sqrt{3})(1 + \tan^2 \theta) = k(1 - 3\tan^2 \theta).$$ [4]

(ii) Hence solve the equation

$$\tan(60° + \theta) + \tan(60° - \theta) = 3\sqrt{3},$$

giving all solutions in the interval $0° \leqslant \theta \leqslant 180°$. [3]

Answer: **(ii)** 16.8°, 163.2°. 33/J11/4

16 (i) By sketching a suitable pair of graphs, show that the equation

$$\cot x = 1 + x^2,$$

Answer: 33/J11/6

17 (i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

31/J11/9

Answers: (ii)(a) ± 0.572,

18 (i) Express $(\surd 6)\cos\theta + (\surd 10)\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the value of α correct to 2 decimal places. [3]

(ii) Hence, in each of the following cases, find the smallest positive angle θ which satisfies the equation

(a) $(\surd 6)\cos\theta + (\surd 10)\sin\theta = -4$, [2]

(b) $(\surd 6)\cos\frac{1}{2}\theta + (\surd 10)\sin\frac{1}{2}\theta = 3$. [4]

Answers: (i) $4\cos(\theta - 52.24^\circ)$; (ii)(a) 232.2°, (b) 21.7°.

33/N10/8

19 Solve the equation

$$\tan(45^\circ - x) = 2\tan x,$$

giving all solutions in the interval $0^\circ < x < 180^\circ$. [5]

Answer: 15.7°, 119.3°.

33/J10/3

20 (i) Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

Answer:

33/N10/7

21 Solve the equation

$$\sin\theta = 2\cos 2\theta + 1,$$

giving all solutions in the interval $0^\circ \leqslant \theta \leqslant 360^\circ$. [6]

Answer: 48.6°, 131.4°, 270°.

31/J10/2

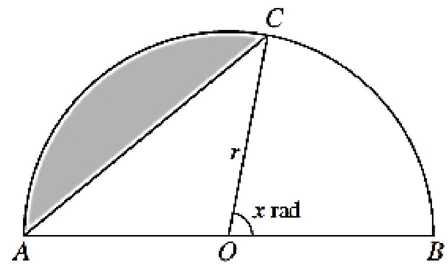
22 (i) Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

$$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

Answers: (i) 1.50, 6.00.

31/J10/3

23



The diagram shows a semicircle ACB with centre O and radius r. The angle BOC is x radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that x satisfies the equation

$$x = \tfrac{3}{4}\pi - \sin x.$$ [3]

Answer: **(iii)** 1.38. 31/J10/6

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24 Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° < \theta < 180°$. [6]

Answers: 18.4°, 135° N16/33/Q3

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25 **(i)** Show that the equation $(\sqrt{2})\operatorname{cosec} x + \cot x = \sqrt{3}$ can be expressed in the form $R\sin(x - \alpha) = \sqrt{2}$, where $R > 0$ and $0° < \alpha < 90°$. [4]

(ii) Hence solve the equation $(\sqrt{2})\operatorname{cosec} x + \cot x = \sqrt{3}$, for $0° < x < 180°$. [4]

Answer: $x = 75°$ and $165°$ N18/33/Q6

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26 By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

Answer: 131.8° J16/31/Q3

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27 **(i)** Express $(\sqrt{5})\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$(\sqrt{5})\cos\tfrac{1}{2}x + 2\sin\tfrac{1}{2}x = 1.2,$$

for $0° < x < 360°$. [3]

Answer: **(i)** $R = 3$ $\alpha = 41.81°$; **(ii)** 216.5° J16/33/Q3

28 (i) By first expanding $2\sin(x-30°)$, express $2\sin(x-30°)-\cos x$ in the form $R\sin(x-\alpha)$, where $R>0$ and $0°<\alpha<90°$. Give the exact value of R and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$2\sin(x-30°)-\cos x=1,$$

for $0°<x<180°$. [3]

Answers: **(i)** $R=\sqrt{7}$, α=49.11° **(ii)** 71.3° **J17/31/Q8**

29 Prove the identity $\dfrac{\cot x-\tan x}{\cot x+\tan x}\equiv\cos 2x$. [3]

J17/33/Q1

30 (i) Given that $\sin(x-60°)=3\cos(x-45°)$, find the exact value of $\tan x$. [4]

(ii) Hence solve the equation $\sin(x-60°)=3\cos(x-45°)$, for $0°<x<360°$. [2]

Answer: **(i)** $\dfrac{3\sqrt{2}+\sqrt{3}}{1-3\sqrt{2}}$ **(ii)** 118.5°, 298.5° **J18/31/Q2**

31 (i) By first expanding $(\cos^2x+\sin^2x)^3$, or otherwise, show that

$$\cos^6x+\sin^6x=1-\tfrac{3}{4}\sin^2 2x.$$ [4]

(ii) Hence solve the equation

$$\cos^6x+\sin^6x=\tfrac{2}{3},$$

for $0°<x<180°$. [4]

Answer: **(ii)** 20.9°, 69.1°, 110.9°, 159.1° **J18/33/Q5**

32 **Throughout this question the use of a calculator is not permitted.**

(i) Express $\cos\theta+2\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. [3]

(ii) Hence, showing all necessary working, show that $\displaystyle\int_0^{\frac{1}{4}\pi}\frac{15}{(\cos\theta+2\sin\theta)^2}\,d\theta=5.$ [5]

CHAPTER: 8: DIFFERENTIATION

AS Revision

Example 1

1 Differentiate with respect to x.

a $y = 5x^3 - \frac{3}{x^2} + 2\sqrt{x}$

b $y = \frac{x^8 - 4x^5 + x^2}{2x^3}$

Example 2

Differentiate with respect to x.

a $(3x - 5)^4$

b $\frac{4}{\sqrt{1 - 2x}}$

Example 3

Find the equation of the normal to the curve $y = x^3 - 5x^2 + 2x - 1$ at the point $(1, -3)$.

Example 4

Find the stationary points on the curve $y = x^3 - 3x^2 + 2$ and determine their nature.

Example 5

The equation of a curve is $y = \frac{1}{6}(2x - 3)^3 - 4x$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the y axis.

(iii) Find the set of values of x for which $\frac{1}{6}(2x - 3)^3 - 4x$ is an increasing function of x.

[Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 June 2010]

Example 6

A curve is such that $\frac{dy}{dx} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. The curve passes through the point $(4, \frac{2}{3})$.

(ii) Find $\frac{d^2y}{dx^2}$. **[2]**

(iii) Find the coordinates of the stationary point and determine its nature. **[5]**

Example 7

The base of a cuboid has sides of length x cm and $3x$ cm. The volume of the cuboid is $288\,\text{cm}^3$.

(i) Show that the total surface area of the cuboid, $A\,\text{cm}^2$, is given by

$$A = 6x^2 + \frac{768}{x}.$$ **[3]**

(ii) Given that x can vary, find the stationary value of A and determine its nature. **[5]**

Example 8

The volume of a spherical balloon is increasing at a constant rate of $50\,\text{cm}^3$ per second. Find the rate of increase of the radius when the radius is 10 cm. [Volume of a sphere = $\frac{4}{3}\pi r^3$.] **[4]**

Answer: 0.0398 11/J11/2

Differentiation of Exponential and Logarithmic Functions

If $y = e^x$, then $\frac{dy}{dx} = e^x$.

If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$.

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)\,e^{f(x)}$.

Example 1

Differentiate

a) e^{7x} **b)** e^{5x-4} **c)** e^{x^3}.

Example 2

Find $\frac{dy}{dx}$ when

a) $y = 4e^{2x}$ **b)** $y = -2e^{\sqrt{x}}$ **c)** $y = 3e^{\frac{1}{x}}$.

Example 3

Find the exact value of the gradient of the tangent to the curve $y = e^x - 6\sqrt{x}$ when $x = 1$.

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.

If $y = \ln[f(x)]$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$.

Example 4

Find $\frac{dy}{dx}$ when **a)** $y = \ln 5x$ **b)** $y = \ln(3x^2 - 2)$.

Example 5

Differentiate

a) $\ln(9x + 2)$ **b)** $\ln(e^x - 3x)$ **c)** $4\ln 8x$ **d)** $\ln(5x - 1)^7$.

Example 6

Find the coordinates of the turning point on the curve $y = x - \ln x$, and determine whether this point is a maximum or a minimum point.

Product Rule

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

Example 7

Find $\frac{dy}{dx}$ when $y = e^{2x}(x^3 - 3)$.

Example 8

Find $\frac{dy}{dx}$ when $y = (2x + 1)(x - 5)^4$.

Example 9

Show that $\dfrac{dy}{dx} = \dfrac{4x(5x+2)}{\sqrt{2x+1}}$ when $y = 4x^2\sqrt{2x+1}$.

Example 10

Find the gradient of the tangent to the curve $y = x^2 e^{-x}$ at the point $\left(1, \dfrac{1}{e}\right)$.

Quotient Rule

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

Example 11

Find $\dfrac{dy}{dx}$ when $y = \dfrac{x^3}{x^2+1}$.

Example 12

Find $\dfrac{dy}{dx}$ when $y = \dfrac{4x+3}{\sqrt{(2x-1)}}$.

Example 13

Show that if $y = \dfrac{7\ln x - x^3}{e^{3x}}$ then $\dfrac{dy}{dx} = \dfrac{7}{e^3}$ when $x = 1$.

Differentiation of Trigonometric functions

If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$.

If $y = \cos x$, then $\dfrac{dy}{dx} = -\sin x$.

If $y = \tan x$, then $\dfrac{dy}{dx} = \sec^2 x$.

Example 14

Find $\frac{dy}{dx}$ when

a) $y = \cos 2x$ **b)** $y = 5\sin 3x^2$ **c)** $y = 7\tan\left(4x + \frac{\pi}{2}\right)$.

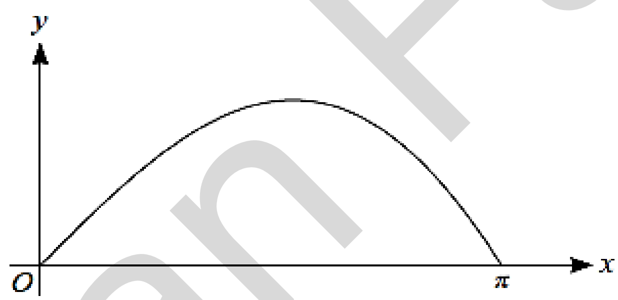
Example 15

Show that the maximum value of the curve $y = x - 2\sin x$ for $0 \leq x \leq 2\pi$ is $\frac{5\pi}{3} + \sqrt{3}$, and find the minimum value of the curve.

Example 16

If $y = \ln\sqrt{1 - \cos x}$, show that $\frac{d^2y}{dx^2} = \frac{1}{2(\cos x - 1)}$.

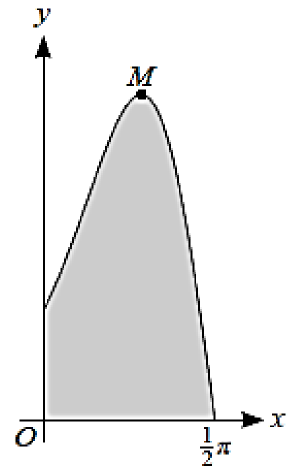
June 2014/32 Question 8a



The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

(i) Find $\frac{dy}{dx}$ and show that $4\frac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

June 2014/33 Q9(ii)



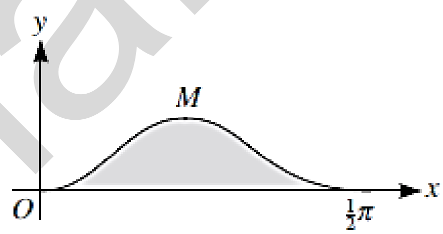
The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(ii) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

June 2013/32 Question 6i

(i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

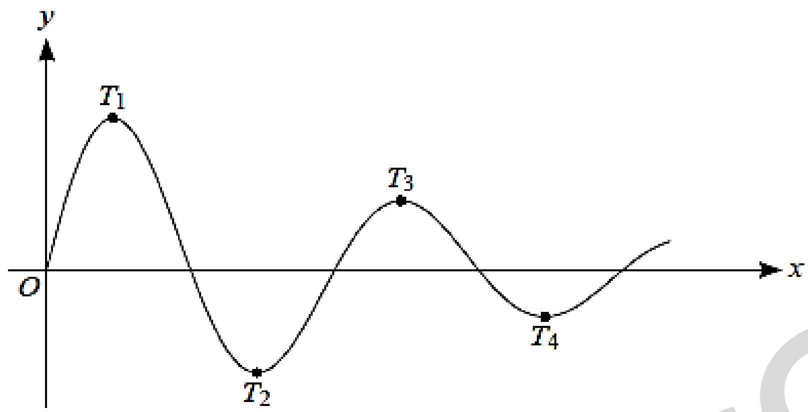
June 2013/33 Question 9



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [6]

June 2014/31 Question 10

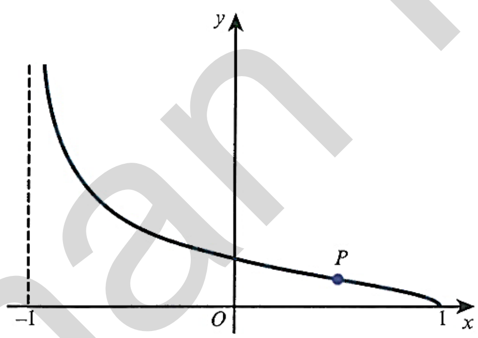


The diagram shows the curve $y = 10e^{-\frac{1}{2}x}\sin 4x$ for $x \geqslant 0$. The stationary points are labelled T_1, T_2, T_3, ... as shown.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

Example 19



The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

i By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. **[5]**

ii The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 June 2010

HOMEWORK: DIFFERENTIATION VARIANT 32

1 The equation of a curve is $y = x \sin 2x$, where x is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

Answer: $y = x$. J07/Q3

2 The curve $y = e^x + 4e^{-2x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether the stationary point is a maximum or a minimum point. [2]

Answers: (i) ln 2; (ii) Minimum point N02/Q4

3 The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Answers: (i) $\frac{1}{2}\ln 2$; (ii) maximum. N06/Q3

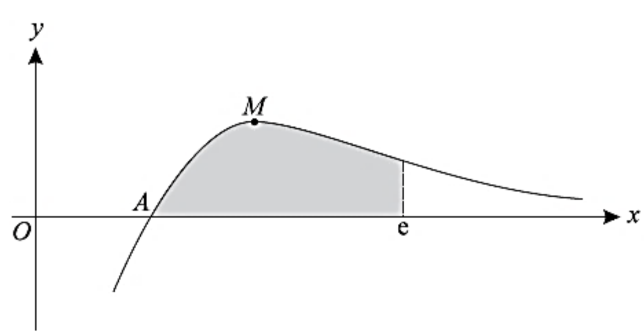
4 The curve $y = \dfrac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the x-coordinate of this point. [5]

Answer: $-\frac{1}{4}\pi$ or -0.785 radians. N08/Q3

5 A curve has equation $y = e^{-3x}\tan x$. Find the x-coordinates of the stationary points on the curve in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Give your answers correct to 3 decimal places. [6]

Answer: 0.365, 1.206. N09/31/Q4

6



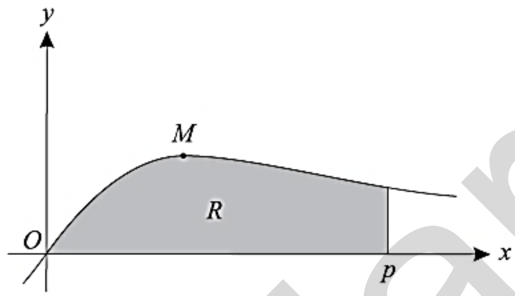
The diagram shows the curve $y = \dfrac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

Answers: **(i)** 1; **(ii)** $x = e^{\frac{1}{2}}$, J04/Q10

7

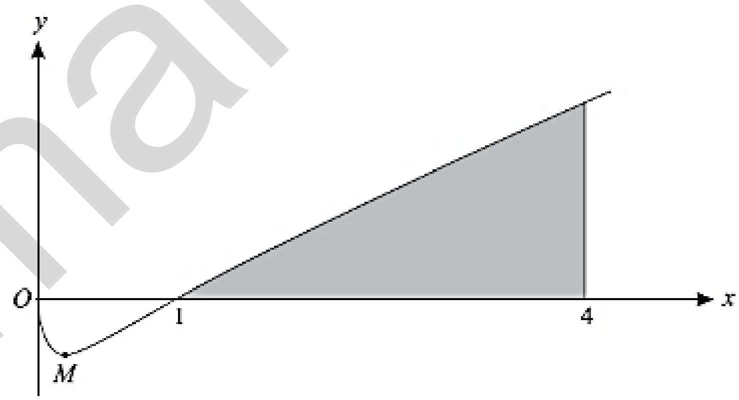


The diagram shows part of the curve $y = \dfrac{x}{x^2+1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.

(i) Calculate the x-coordinate of M. [4]

Answers: **(i)** 1; J05/Q9

8

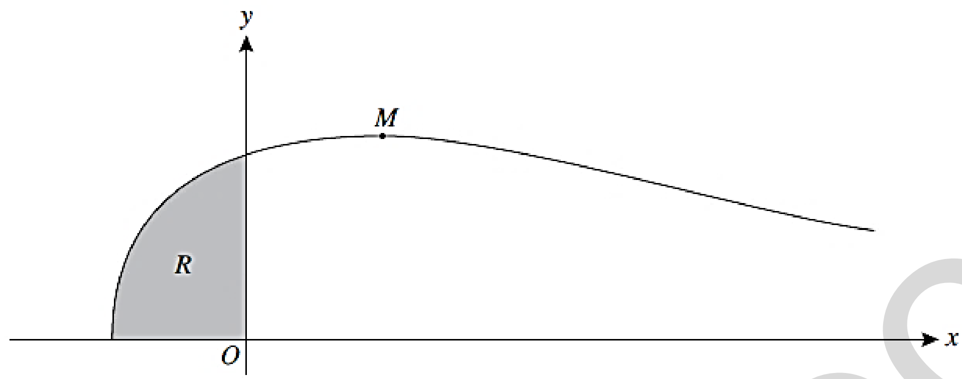


The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

(i) Find the exact value of the x-coordinate of M. [4]

Answers: **(i)** e^{-2}; **(ii)** 4.28. J06/Q8

9



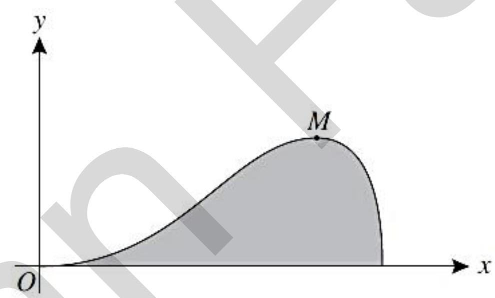
The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

(i) Find the x-coordinate of M. [4]

J08/Q9

Answers: (i) $\frac{1}{2}$;

10



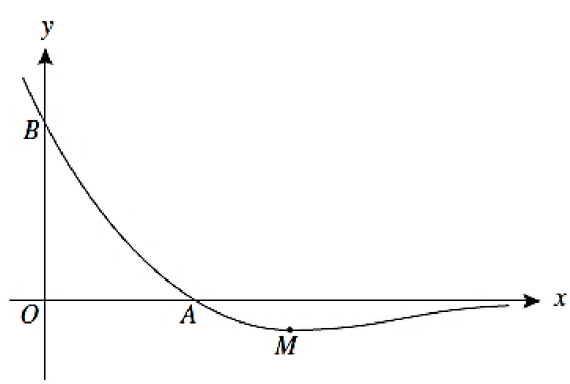
The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

J09/Q10

Answers: (i)

11

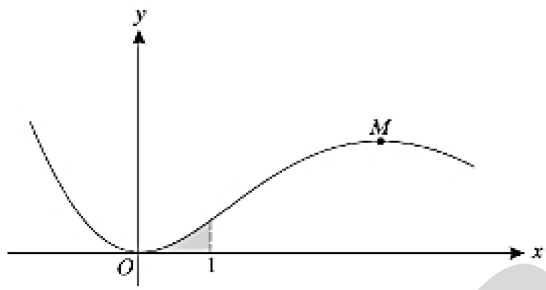


The diagram shows the curve $y = (3 - x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

(i) Calculate the x-coordinate of M. [4]

Answers: (i) $3\frac{1}{2}$ N03/Q6

12

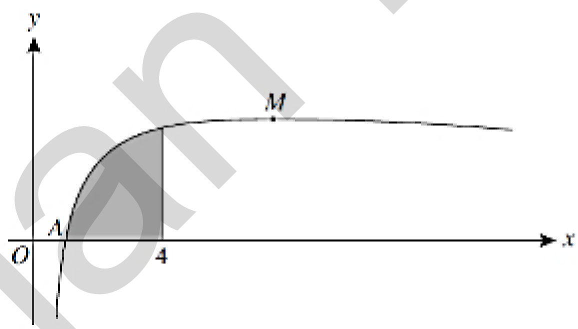


The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the x-coordinate of M, the maximum point of the curve. [4]

Answers: (i) 4; N04/Q7

13



The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.

(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

Answers: (i) (1, 0); (ii) e^2. N09/31/Q9

14 The equation of a curve is $y = x + \cos 2x$. Find the x-coordinates of the stationary points of the curve for which $0 \leqslant x \leqslant \pi$, and determine the nature of each of these stationary points. [7]

Answers: $\frac{1}{12}\pi$, maximum; $\frac{5}{12}\pi$, minimum. N05/Q3

15 The equation of a curve is $y = 3\sin x + 4\cos^3 x$.

(i) Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. [6]

(ii) Determine the nature of the stationary point in this interval for which x is least. [2]

– *Answer:* **(i)** $\frac{1}{12}\pi, \frac{5}{12}\pi, \frac{1}{2}\pi$; **(ii)** Maximum J12/32/Q6

16 **(i)** The polynomial $f(x)$ is of the form $(x-2)^2 g(x)$, where $g(x)$ is another polynomial. Show that $(x-2)$ is a factor of $f'(x)$. [2]

(ii) The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where a and b are constants, has a factor $(x-2)^2$. Using the factor theorem and the result of part **(i)**, or otherwise, find the values of a and b. [5]

Answer: **(ii)** $a = -4$, $b = 3$ J14/32/Q5

17 A curve has equation $y = \cos x \cos 2x$. Find the x-coordinate of the stationary point on the curve in the interval $0 < x < \frac{1}{2}\pi$, giving your answer correct to 3 significant figures. [6]

Answer: 1.15 J15/32/Q3

18 The equation of a curve is $y = \dfrac{1+x}{1+2x}$ for $x > -\frac{1}{2}$. Show that the gradient of the curve is always negative. [3]

N13/32/Q1

19 The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\dfrac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

Answer: $e^{-2x}(-1+\tan x)^2$; $\frac{1}{4}\pi$ N15/32/Q5

20 A curve has equation

$$y = \frac{2 - \tan x}{1 + \tan x}.$$

Find the equation of the tangent to the curve at the point for which $x = \frac{1}{4}\pi$, giving the answer in the form $y = mx + c$ where c is correct to 3 significant figures. [6]

Answer: $y = -1.5x + 1.68$ N15/33/Q3

HOMEWORK: DIFFERENTIATION– VARIANTS 31 & 33

1 The curve with equation $y = \frac{e^{2x}}{4 + e^{3x}}$ has one stationary point. Find the exact values of the coordinates of this point. [6]

Answer: $(\ln 2, \frac{1}{3})$ 33/J15/4

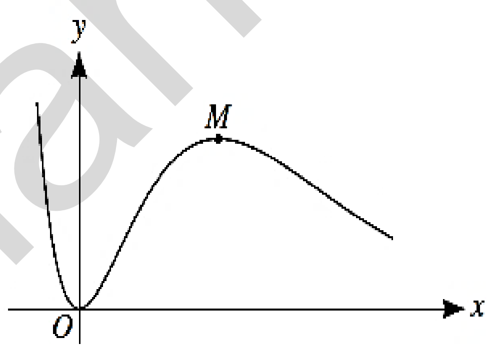
2 The equation of a curve is

$$y = 3\cos 2x + 7\sin x + 2.$$

Find the x-coordinates of the stationary points in the interval $0 \leqslant x \leqslant \pi$. Give each answer correct to 3 significant figures. [7]

Answers: 0.623, 2.52, 1.57 ($\frac{1}{2}\pi$ allowed) 31/J15/4

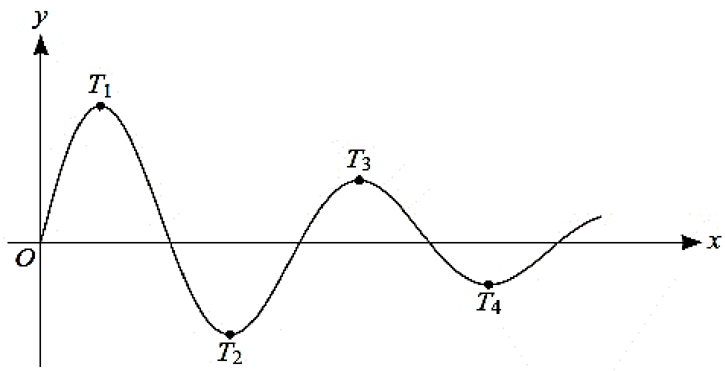
3



The diagram shows the curve $y = x^2 e^{2-x}$ and its maximum point M

Answer 31/J15/9

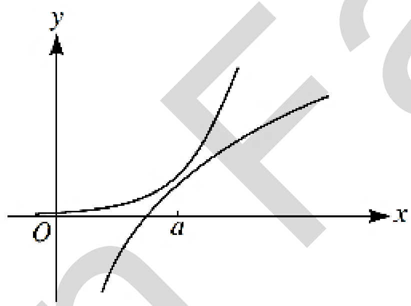
4



(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

Answers: **(i)** 0.362, 1.147 **(ii)** 33 — 31/J14/10

5

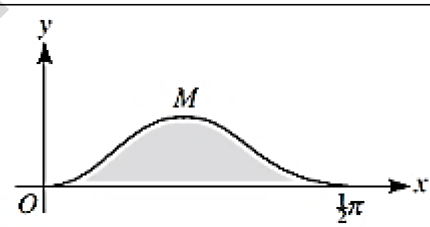


The diagram shows the curves $y = e^{2x-3}$ and $y = 2\ln x$. When $x = a$ the tangents to the curves are parallel.

(i) Show that a satisfies the equation $a = \frac{1}{2}(3 - \ln a)$. [3]

Answer: — 33/J13/6

6



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [6]

Answer: 0.685 — 33/J13/9

7 For each of the following curves, find the gradient at the point where the curve crosses the y-axis:

(i) $y = \dfrac{1+x^2}{1+e^{2x}}$; [3]

31/J13/5

Answer: $-\dfrac{1}{2}$

8 The expression $f(x)$ is defined by $f(x) = 3xe^{-2x}$.

(i) Find the exact value of $f'(-\frac{1}{2})$. [3]

33/N12/5

Answers: (i) $6e$

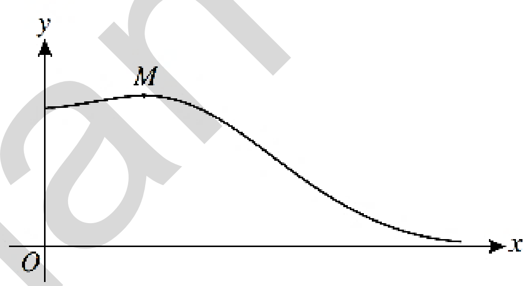
9 (i) By differentiating $\dfrac{1}{\cos x}$, show that if $y = \sec x$ then $\dfrac{dy}{dx} = \sec x \tan x$. [2]

(ii) Show that $\dfrac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\dfrac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

Answer: 31/N12/5

10



The diagram shows the curve $y = e^{-\frac{1}{2}x^2}\sqrt{(1+2x^2)}$ for $x \geqslant 0$, and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

31/N12/8

Answers: (i) $\dfrac{1}{\sqrt{2}}$

11 The curve with equation $y = \dfrac{e^{2x}}{x^3}$ has one stationary point.

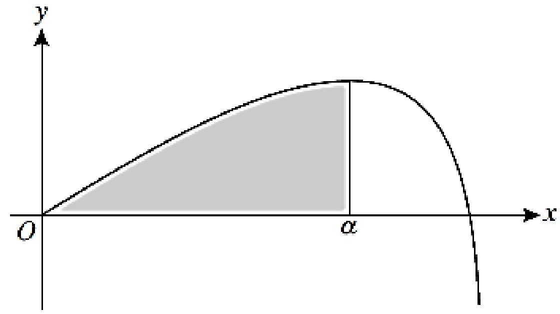
(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

33/J12/4

Answers: (i) $x = \dfrac{3}{2}$ (ii) Minimum point

12



The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The x-coordinate of the maximum point is α and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

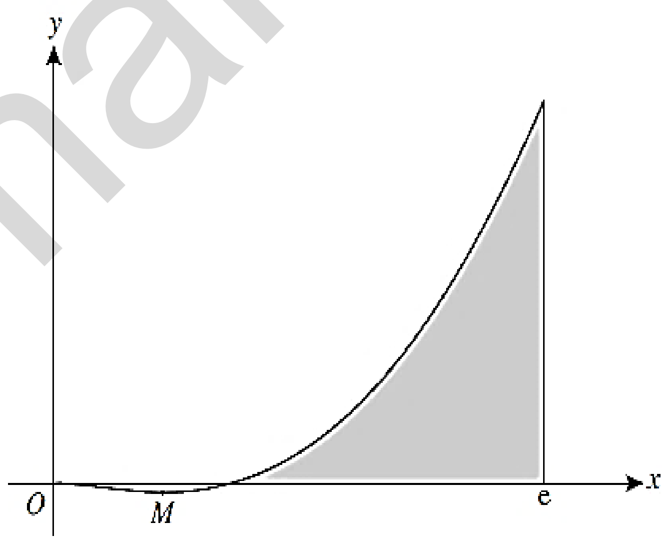
(i) Show that $\alpha = \tfrac{2}{3}\pi$. [3]

Answer: 31/J12/5

13 The equation of a curve is $y = \dfrac{e^{2x}}{1 + e^{2x}}$. Show that the gradient of the curve at the point for which $x = \ln 3$ is $\frac{9}{50}$. [4]

33/N11/2

14



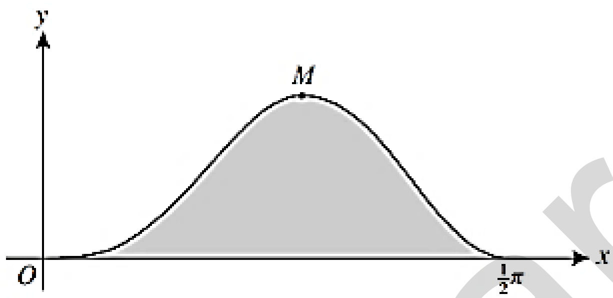
The diagram shows the curve $y = x^2 \ln x$ and its minimum point M.

Answer: (i) $x = \frac{1}{\sqrt{e}}$, $y = -\frac{1}{2e}$ 31/N11/9

15 The curve $y = \frac{\ln x}{x^3}$ has one stationary point. Find the x-coordinate of this point. [4]

Answer: $e^{\frac{1}{3}}$ or 1.40. 33/J11/2

16



The diagram shows the curve $y = 5\sin^3 x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [5]

Answers: (i) 0.886; 33/J11/8

17 Find $\frac{dy}{dx}$ in each of the following cases:

(i) $y = \ln(1 + \sin 2x)$, [2]

(ii) $y = \frac{\tan x}{x}$. [2]

Answers: (i) $\frac{2\cos 2x}{1+\sin 2x}$; (ii) $\frac{x\sec^2 x - \tan x}{x^2}$. 31/J11/2

18 The curve with equation

$$6e^{2x} + ke^y + e^{2y} = c,$$

where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

(i) Show that $58 + 2k = c$. [2]

(ii) Given also that the gradient of the curve at P is -6, find the values of k and c. [5]

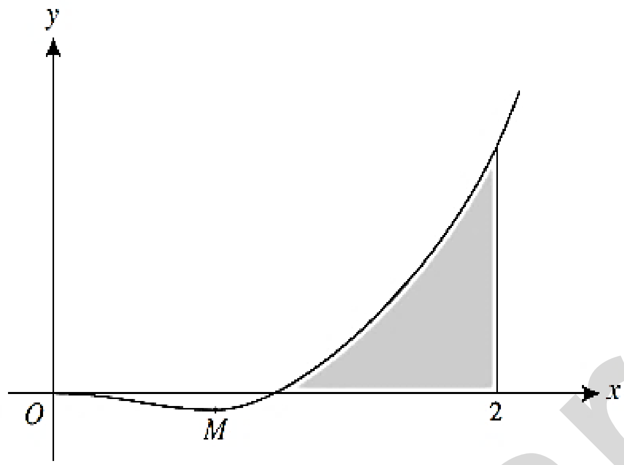
Answers: (ii) 5, 68. 31/J11/5

19 It is given that $f(x) = 4\cos^2 3x$.

Answers: **(i)** $-6\sqrt{3}$;

33/N10/4

20

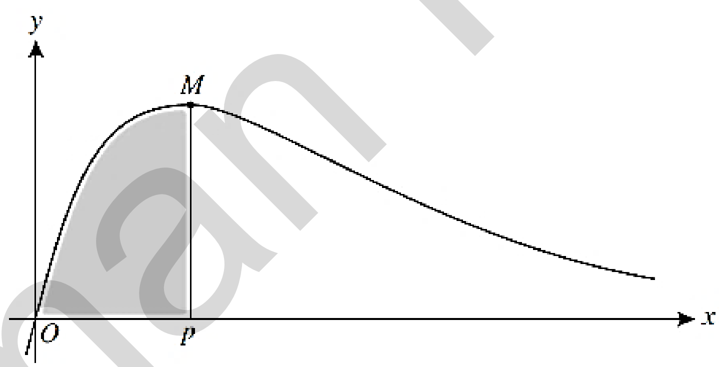


The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

Answers: **(i)** $(e^{-\frac{1}{3}}, -\frac{1}{3e})$

31/N10/9

21



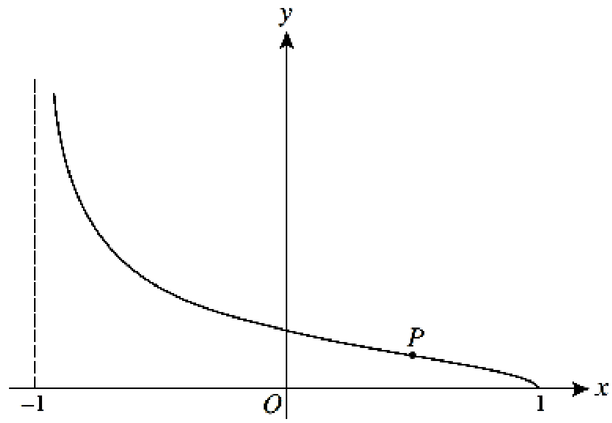
The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. [4]

Answer: **(i)** ln2.

33/J10/5

22



The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

(i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x. Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. [5]

(ii) The gradient of the normal to the curve has its maximum value at the point P shown in the diagram. Find, by differentiation, the x-coordinate of P. [4]

Answer: **(ii)** $\frac{1}{2}$. **J10/31/Q9**

23 The equation of a curve is $y = \frac{\sin x}{1 + \cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. [4]

Answer: $y = \frac{1}{1 + \cos x}$ **N16/33/Q2**

24 The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

Answer: 0.421 **J16/31/Q5**

25 A curve has equation $y = \frac{2}{3}\ln(1 + 3\cos^2 x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan x$. [4]

7

(ii) Hence find the x-coordinate of the point on the curve where the gradient is -1. Give your answer correct to 3 significant figures. [2]

Answers: **(i)** $dy/dx = -4\tan x/(4 + \tan^2 x)$ **(ii)** $x = 1.11$ **J17/33/Q5**

26 A curve has equation $y = \dfrac{e^{3x}}{\tan \frac{1}{2}x}$. Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]

Answer: 0.340 2.802

J18/31/Q3

CHAPTER 9: IMPLICIT DIFFERENTIATION

Functions of the form $y = f(x)$ are **explicit functions** as y is given explicitly in terms of x.

However, some functions that have two variables are not of this form – that is, sometimes we are not given one variable explicitly in terms of the other. We call these functions **implicit functions**.

Examples of implicit functions include:

(1) $x^2 + 3xy^2 - 3y = 0$

(2) $x \ln(y+2) - x^2 = 2y$

(3) $\dfrac{4y^2}{y^2 + x} = 5$

> We use **implicit differentiation** when one of the variables is not given explicitly as a function of the other variable.

Example 1

Find $\dfrac{dy}{dx}$ in terms of x and y when $x^2 - 3y^3 = 4y^2$.

Example 2

Example 13.6 *Find the coordinates of the stationary points of the curve* $x^2 + xy + y^2 = 27$.

Example 3

Example 13.5 *Find the gradient of the curve* $x^3 - 5xy^2 + 2 = 0$ *at the point* $(2, -1)$.

Example 4

Example 13.7 *The equation of a curve is given by* $2x - 1 = x^2 \ln y$.

(i) *Show that* $\dfrac{dy}{dx} = \dfrac{-2y(x-1)}{x^3}$.

(ii) *Find the equation of the tangent to the curve at the point where* $y = 1$.

Example 5

Find the equation of the normal to the curve $5x^2 + 6xy - y^2 = 10$ at the point $(1, 5)$.

Example 6

The equation of a curve is $\ln(xy) - y^3 = 1$.

i Show that $\dfrac{dy}{dx} = \dfrac{y}{x(3y^3 - 1)}$. **[4]**

ii Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q7 November 2012

Example 7

The curve with equation $6e^{2x} + ke^{y} + e^{2y} = c$, where k and c are constants, passes through the point P with coordinates $(\ln 3, \ln 2)$.

i Show that $58 + 2k = c$. **[2]**

ii Given also that the gradient of the curve at P is -6, find the values of k and c. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 June 2011

HOMEWORK: IMPLICIT DIFFERENTIATION VARIANT 32

1 Find the gradient of the curve with equation

$$2x^2 - 4xy + 3y^2 = 3,$$

at the point (2, 1). [4]

Answer: 2. J04/Q3

2 The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

Answer: (ii) $y + 4x + 1 = 0$. J10/32/Q6

3 The equation of a curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a},$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x and y. [3]

(ii) The straight line with equation $y = x$ intersects the curve at the point P. Find the equation of the tangent to the curve at P. [3]

Answers: (i) $-\sqrt{\frac{y}{x}}$; (ii) $x + y = \frac{1}{2}a$. N03/Q4

4 The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$. [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

Answer: (ii) (1, 1). N06/Q6

5 The curve with equation $y = e^{-x} \sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

Answers: (i) $\frac{1}{4}\pi$ or 0.785 radians; (ii) Maximum. N07/Q4

6 The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

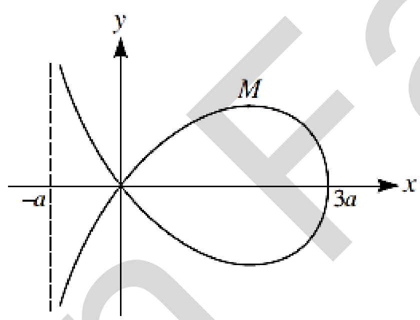
(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

Answers: (i) $\frac{3x^2 - 2xy}{x^2 + 3y^2}$; (ii) $8x - 7y - 9 = 0$. N09/32/Q3

7 The equation of a curve is $xy(x + y) = 2a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [8]

Answer: $(a, -2a)$. J08/Q6

8



The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

Answer: $\sqrt{3}a$ J13/32/Q5

9 The equation of a curve is $\ln(xy) - y^3 = 1$.

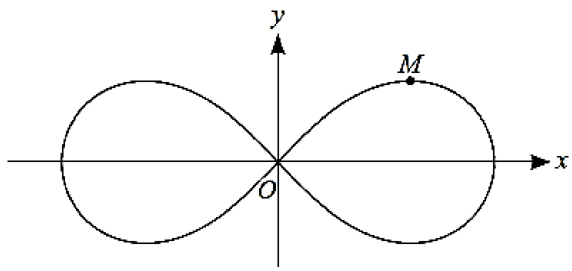
(i) Show that $\frac{dy}{dx} = \frac{y}{x(3y^3 - 1)}$. [4]

(ii) Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. [4]

Answer: **(ii)** (5.47, 0.693). N12/32/Q7

HOMEWORK: IMPLICIT DIFFERENTIATION–VARIANTS 31 & 33

1



The diagram shows the curve $(x^2 + y^2)^2 = 2(x^2 - y^2)$ and one of its maximum points M. Find the coordinates of M. [7]

33/J14/6

Answer: $\left(\frac{1}{2}\sqrt{3}, \frac{1}{2}\right)$

2 A curve has equation $3e^{2x}y + e^x y^3 = 14$. Find the gradient of the curve at the point $(0, 2)$. [5]

33/N13/4

Answer: $-\frac{4}{3}$

3 For each of the following curves, find the gradient at the point where the curve crosses the y-axis:

(i) $y = \dfrac{1 + x^2}{1 + e^{2x}}$; [3]

(ii) $2x^3 + 5xy + y^3 = 8$. [4]

31/J13/5

Answer: $-\frac{1}{2}$ *Answer*: $-\frac{5}{6}$

4 The equation of a curve is $\ln(xy) - y^3 = 1$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y}{x(3y^3 - 1)}$. [4]

(ii) Find the coordinates of the point where the tangent to the curve is parallel to the y-axis, giving each coordinate correct to 3 significant figures. [4]

31/N12/7

Answer: **(ii)** (5.47, 0.693).

5 The equation of a curve is $3x^2 - 4xy + y^2 = 45$.

(i) Find the gradient of the curve at the point $(2, -3)$. [4]

(ii) Show that there are no points on the curve at which the gradient is 1. [3]

31/J12/6

Answer: (i) 12/7

6 The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

Answer: **(ii)** $(-2,-1)$ $(0,\sqrt[3]{3})$ J16/31/Q7

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7 The equation of a curve is $2x^3 - y^3 - 3xy^2 = 2a^3$, where a is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x^2 - y^2}{y^2 + 2xy}$. [4]

(ii) Find the coordinates of the two points on the curve at which the tangent is parallel to the y-axis. [5]

Answer: **(ii)** $(a, 0)$, $(-a, 2a)$ J18/33/Q8

CHAPTER 10: DIFFERENTIATION OF PARAMETRIC EQUATIONS

When x and y are related via a third variable, t, then t is called a **parameter**.
Equations which state x and y in terms of t are called **parametric equations**.

Suppose $x = t^3$ and $y = t^2$.
We can look at values of x and y for particular values of t.

When differentiating parametric equations involving a parameter t,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

This is called **parametric differentiation**.

Example 1

a) If $x = 2t^3$ and $y = 5t^2 - 6$, find $\frac{dy}{dx}$ in terms of t.

b) Find the value of $\frac{dy}{dx}$ when $x = -16$.

Example 2

The parametric equations of a curve are $x = 2 + t$, $y = \frac{1}{t^2} - 6$. Show that $\frac{dy}{dx} = -2t^{-3}$.

When differentiating parametric equations where the parameter is an angle, θ,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}, \text{ where } \theta \text{ is in radians}$$

Example 3

The parametric equations of a curve are given by $x = 2\cos\theta$, $y = \sin^2\theta$, $0 \leq \theta \leq 2\pi$.
Find the values of θ when the tangent to the curve is parallel to the x-axis.

Example 4

Show that $\frac{dy}{dx} = \frac{\sin\theta}{1 + \cos\theta}$ when $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, where a is a constant.

Example 5

The parametric equations of a curve are $x = 1 + 2\sin^2\theta,\ y = 4\tan\theta$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

a Find $\frac{dy}{dx}$ in terms of the parameter θ.

b Find the coordinates of the point on the curve where the tangent is parallel to the y-axis.

Example 6

The parametric equations of a curve are

$x = 1 + \ln(t-2),\ y = t + \frac{9}{t}$, for $t > 2$.

i Show that $\frac{dy}{dx} = \frac{(t^2-9)(t-2)}{t^2}$. **[3]**

ii Find the coordinates of the only point on the curve at which the gradient is equal to 0. **[3]**

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Example 7

Find the value of $\frac{dy}{dx}$ when $x = 4$ in each of the following cases:

i $y = x\ln(x-3)$, **[4]**

ii $y = \frac{x-1}{x+1}$. **[3]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q5 June 2011

Example 8

The parametric equations of a curve are $x = e^{3t},\ y = t^2e^t + 3$.

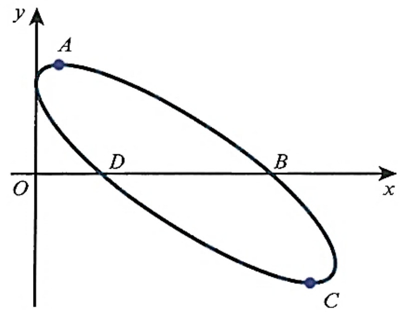
i Show that $\frac{dy}{dx} = \frac{t(t+2)}{3e^{2t}}$. **[4]**

ii Show that the tangent to the curve at the point (1, 3) is parallel to the x-axis. **[2]**

iii Find the exact coordinates of the other point on the curve at which the tangent is parallel to the x-axis. **[2]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q7 November 2011

Example 9



The parametric equations of a curve are $x = 6\sin^2 t$, $y = 2\sin 2t + 3\cos 2t$, for $0 \leqslant t < \pi$. The curve crosses the x-axis at the points B and D and the stationary points are A and C, as shown in the diagram.

i Show that $\dfrac{dy}{dx} = \dfrac{2}{3}\cot 2t - 1$. **[5]**

ii Find the values of t at A and C, giving each answer correct to 3 decimal places. **[3]**

iii Find the value of the gradient of the curve at B. **[3]**

Example 10

June 2013/31 Question 5

For each of the following curves, find the gradient at the point where the curve crosses the y-axis:

(i) $y = \dfrac{1 + x^2}{1 + e^{2x}}$; [3]

(ii) $2x^3 + 5xy + y^3 = 8$. [4]

Example 11

November 2014/31 Question 4

The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x\sin t - \tan t$. [3]

Example 12

November 2014/33 Question 2

A curve is defined for $0 < \theta < \frac{1}{2}\pi$ by the parametric equations

$$x = \tan\theta, \quad y = 2\cos^2\theta\sin\theta.$$

Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 4\cos^3\theta$. [5]

Example 13

June 2014/31 Question 3

The parametric equations of a curve are

$$x = \ln(2t+3), \quad y = \frac{3t+2}{2t+3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. [6]

HOMEWORK: DIFFERENTIATION OF PARAMETRIC EQUATIONS VARIANT 32

1 The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \quad y = 1 - \cos 2\theta.$$

Show that $\frac{dy}{dx} = \tan\theta$. [5]

J06/Q3

2 The parametric equations of a curve are

$$x = a\cos^3 t, \quad y = a\sin^3 t,$$

where a is a positive constant and $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin t + y\cos t = a\sin t\cos t.$$ [3]

(iii) Hence show that, if this tangent meets the x-axis at X and the y-axis at Y, then the length of XY is always equal to a. [2]

Answer: (i) $-\tan t$. J09/Q6

3 The parametric equations of a curve are

$$x = a(2\theta - \sin 2\theta), \quad y = a(1 - \cos 2\theta).$$

Show that $\frac{dy}{dx} = \cot\theta$. [5]

N08/Q4

4 The parametric equations of a curve are

$$x = \ln(\tan t), \quad y = \sin^2 t,$$

where $0 < t < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of t. [4]

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. [3]

Answer: (i) $2\sin^2 t\cos^2 t$; (ii) $y = \frac{1}{2}x + \frac{1}{2}$. J11/32/Q5

5 The parametric equations of a curve are

$$x = t - \tan t, \quad y = \ln(\cos t),$$

for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \cot t$. [5]

(ii) Hence find the x-coordinate of the point on the curve at which the gradient is equal to 2. Give your answer correct to 3 significant figures. [2]

Answer. **(ii)** – 0.0364 J14/32/Q4

6 The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5]

Answer. –cos *t* N11/32/Q2

7 The parametric equations of a curve are

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t.$$

Show that $\frac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$. [6]

N13/32/Q4

8 The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x\sin t - \tan t$. [3]

N14/32/Q4

HOMEWORK: DIFFERENTIATION OF PARAMETRIC EQUATIONS– VARIANTS 31 & 33

1 The parametric equations of a curve are

$$x = a\cos^4 t, \quad y = a\sin^4 t,$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of t. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter t is

$$x\sin^2 t + y\cos^2 t = a\sin^2 t\cos^2 t.$$ [3]

(iii) Hence show that if the tangent meets the x-axis at P and the y-axis at Q, then

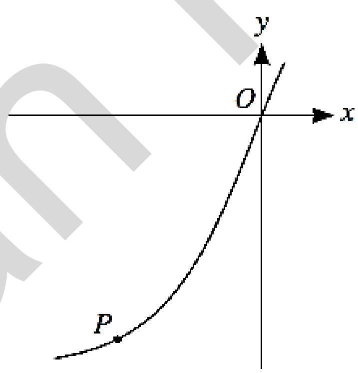
$$OP + OQ = a,$$

where O is the origin. [2]

Answers: **(i)** $-\tan^2 t$

33/J15/5

2



The diagram shows part of the curve with parametric equations

$$x = 2\ln(t + 2), \qquad y = t^3 + 2t + 3.$$

(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is $\frac{1}{2}$.

(a) Show that $p = \frac{1}{3p^2 + 2} - 2$. [1]

Answer: **(i)** $\frac{5}{2}$ **(ii)(b)** $(-5.15, -7.97)$

31/J15/10

3 A curve is defined for $0 < \theta < \frac{1}{2}\pi$ by the parametric equations

$$x = \tan\theta, \quad y = 2\cos^2\theta\sin\theta.$$

Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 4\cos^3\theta$. [5

33/N14/2

4 The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \sin t$. [4

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t i $y = x\sin t - \tan t$. [3

Answer: 31/N14/4

5 The parametric equations of a curve are

$$x = \ln(2t+3), \quad y = \frac{3t+2}{2t+3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. [6]

Answer: $\dfrac{5}{2}$ 31/J14/3

6 The parametric equations of a curve are

$$x = \frac{4t}{2t+3}, \quad y = 2\ln(2t+3).$$

(i) Express $\dfrac{dy}{dx}$ in terms of t, simplifying your answer. [4

(ii) Find the gradient of the curve at the point for which $x = 1$. [2

Answers: (i) $\dfrac{1}{3}(2t+3)$; (ii) 2. 33/N12/3

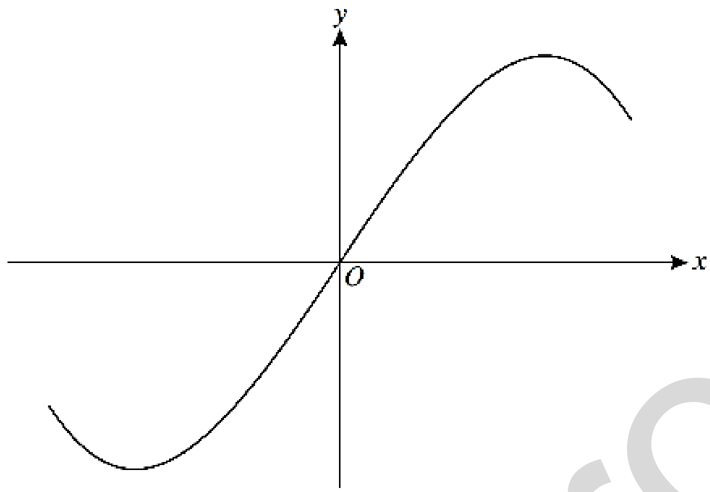
7 The parametric equations of a curve are

$$x = \sin 2\theta - \theta, \quad y = \cos 2\theta + 2\sin\theta.$$

Show that $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{1+2\sin\theta}$. [5

Answer: 33/J12/3

8



The diagram shows the curve with parametric equations

$$x = \sin t + \cos t, \quad y = \sin^3 t + \cos^3 t,$$

for $\frac{1}{4}\pi < t < \frac{5}{4}\pi$.

(i) Show that $\frac{dy}{dx} = -3\sin t \cos t$. [3

(ii) Find the gradient of the curve at the origin. [2

(iii) Find the values of t for which the gradient of the curve is 1, giving your answers correct t 2 significant figures. [4

Answers: (i) $-3\sin t\cos t$ (given); (ii) $\frac{3}{2}$; (iii) 1.9, 2.8

33/N11/8

9 The parametric equations of a curve are

$$x = 3(1 + \sin^2 t), \quad y = 2\cos^3 t.$$

Find $\frac{dy}{dx}$ in terms of t, simplifying your answer as far as possible. [5

Answer: $-\cos t$

31/N11/2

10 The parametric equations of a curve are

$$x = \frac{t}{2t+3}, \qquad y = e^{-2t}.$$

Find the gradient of the curve at the point for which $t = 0$. [5]

Answer: -6.

33/N10/2

11 The parametric equations of a curve are

$$x = 2\sin\theta + \sin 2\theta, \quad y = 2\cos\theta + \cos 2\theta,$$

where $0 < \theta < \pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ in terms of θ. [3]

(ii) Hence find the exact coordinates of the point on the curve at which the tangent is parallel to the y-axis. [4]

Answers: **(i)** $-\frac{(2\sin\theta + 2\sin 2\theta)}{(2\cos\theta + 2\cos 2\theta)}$ **(ii)** $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$ **N18/33/Q4**

12 The parametric equations of a curve are

$$x = t + \cos t, \quad y = \ln(1 + \sin t),$$

where $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sec t$. [5]

(ii) Hence find the x-coordinates of the points on the curve at which the gradient is equal to 3. Give your answers correct to 3 significant figures. [3]

Answer: **(ii)** 1.56, −0.898 **J16/33/Q4**

13 The parametric equations of a curve are

$$x = \ln\cos\theta, \quad y = 3\theta - \tan\theta,$$

where $0 \leqslant \theta < \frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $\tan\theta$. [5]

(ii) Find the exact y-coordinate of the point on the curve at which the gradient of the normal is equal to 1. [3]

Answers: **(i)** $\frac{\tan^2\theta - 2}{\tan\theta}$ **(ii)** $\frac{3}{4}\pi - 1$ **J17/31/Q4**

v

CHAPTER 11: INTEGRATION

Integration of Exponentials

Example 1

Find $\int e^{2x+3}\,dx$.

Example 2

Find:

a $\int e^{3x}\,dx$ b $\int e^{-2x}\,dx$ c $\int e^{5x-4}\,dx$

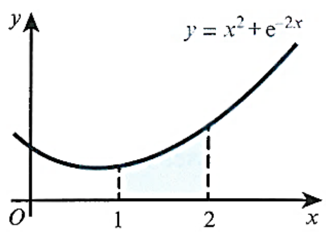
Example 3

Calculate $\int_0^1 e^{2x+3}\,dx$.

Example 4

Evaluate $\int_1^2 9e^{3x-1}\,dx$.

Example 5



Find the area of the shaded region.

Example 6

Example 5

Find the area bounded by the graph of $y = e^{2x}$, the x-axis, the y-axis, and the line $x = 1$.

Integration of $\frac{1}{ax+b}$

Example 7

Example 6

Find $\int \frac{2}{2x-3}\,dx$ and state the values of x for which the answer is valid.

Example 8

Find each of these integrals and state the values of x for which the integral is valid.

a $\int \frac{2}{x}\,dx$ **b** $\int \frac{4}{2x+1}\,dx$ **c** $\int \frac{6}{2-3x}\,dx$

Example 9

Calculate $\int_3^5 \frac{1}{3x-2}\,dx$, giving your answer as a single logarithm (i.e. in the form $\ln k$, where k is real).

Example 10

Find the value of $\int_2^3 \frac{6}{2-3x}\,dx$.

Example 11

Example 11

Calculate $\int_{-2}^{1} \frac{1}{7-2x}\,dx$, giving your answer as a single logarithm (i.e. in the form $\ln k$, where k is real).

Integration of trigonometric functions

Example 12

Find $\int \cos 2x\,dx$.

Example 13

Find $\int \sin\left(3x+\frac{\pi}{4}\right)dx$.

Example 14

Find:

a $\int \sin 2x \, dx$ b $\int \cos 3x \, dx$ c $\int 4\sec^2 \frac{x}{2} \, dx$

Example 15

Find the exact value of $\int_0^{\frac{\pi}{4}} (3 - 2\sin 2x) \, dx$.

Example 16

Calculate the exact value of $\int_0^{\frac{\pi}{6}} 4\tan^2 \theta \, d\theta$.

Example 17

Find the exact value of $\int_0^{\frac{\pi}{4}} 5\tan^2 x \, dx$.

Example 18

a) By writing $2x = 3x - x$ and $4x = 3x + x$, **show that** $\sin 3x \cos x = \frac{1}{2}(\sin 2x + \sin 4x)$.

b) Hence calculate $\int_0^{\frac{\pi}{6}} \sin 3x \cos x \, dx$.

Example 19

a) Show that $(3\sin x - \cos x)^2 = 5 - 3\sin 2x - 4\cos 2x$.

b) Hence calculate $\int_0^{\frac{\pi}{2}} (3\sin x - \cos x)^2 \, dx$.

Example 20

a Show that if $y = \sec x$ then $\frac{dy}{dx} = \tan x \sec x$.

b Hence find the exact value of $\int_0^{\frac{1}{4}\pi} (\cos 2x + 5\tan x \sec x) \, dx$.

Example 21

a Prove that $2\operatorname{cosec} 2x \tan x \equiv \sec^2 x$.

b Hence find the exact value of $\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} (5 + \operatorname{cosec} 2x \tan x) \, dx$.

Example 22

Find the integrals

a) $\int \cos^2 x \, dx$ **b)** $\int 12 \sin^2 x \, dx.$

Example 23

a) Show that $\frac{\cot^2\theta}{1+\cot^2\theta} = \cos^2\theta.$ **b)** Hence calculate $\int_0^{\frac{\pi}{3}} \left(\frac{\cot^2\theta}{1+\cot^2\theta}\right) d\theta.$

Example 24

i Show that $12\sin^2 x\cos^2 x \equiv \frac{3}{2}(1-\cos 4x).$ **[3]**

ii Hence show that $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} 12\sin^2 x\cos^2 x \, dx = \frac{\pi}{8} + \frac{3\sqrt{3}}{16}.$ **[3]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q3 June 2013

Example 25

a Find $\int 4e^x(3+e^{2x})\, dx.$ **[3]**

b Show that $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} (3+2\tan^2\theta)\, d\theta = \frac{1}{2}(8+\pi).$ **[4]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q6 June 2011

Example 26

i Show that $(2\sin x+\cos x)^2$ can be written in the form $\frac{5}{2} + 2\sin 2x - \frac{3}{2}\cos 2x.$ **[5]**

ii Hence find the exact value of $\int_0^{\frac{1}{4}\pi} (2\sin x+\cos x)^2 \, dx.$ **[4]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q7 June 2012

Example 27

i By differentiating $\frac{\cos x}{\sin x}$, show that if $y = \cot x$ then $\frac{dy}{dx} = -\operatorname{cosec}^2 x$. **[3]**

ii By expressing $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$ and using the result of **part i**, show that $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x \, dx = 1 - \frac{1}{4}\pi$. **[4]**

iii Express $\cos 2x$ in terms of $\sin^2 x$ and hence show that $\frac{1}{1 - \cos 2x}$ can be expressed as $\frac{1}{2}\operatorname{cosec}^2 x$.

Hence, using the result of **part i**, find $\int \frac{1}{1 - \cos 2x} \, dx$. **[3]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q8 June 2010

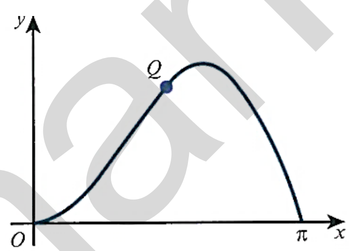
Example 28

i Find $\int \frac{1 + \cos^4 2x}{\cos^2 2x} \, dx$. **[3]**

ii Without using a calculator, find the exact value of $\int_4^{14} \left(2 + \frac{6}{3x - 2}\right) dx$, giving your answer in the form $\ln(ae^b)$, where a and b are integers. **[5]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q7 June 2016

Example 29



The diagram shows the curve $y = x \sin x$, for $0 \leqslant x \leqslant \pi$. The point $Q\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ lies on the curve.

i Show that the normal to the curve at Q passes through the point $(\pi, 0)$. **[5]**

ii Find $\frac{d}{dx}(\sin x - x \cos x)$. **[2]**

iii Hence evaluate $\int_0^{\frac{1}{2}\pi} x \sin x \, dx$. **[3]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q8 November 2010

Example 30

i Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. **[3]**

ii Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2}\ln 3$. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 November 2013

Example 31

i Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. **[4]**

ii Hence show that $\int_{0}^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 November 2016

Example 32

i Using the expansion of $\cos(3x - x)$ and $\cos(3x + x)$, prove that $\frac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x$. **[3]**

ii Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x \, dx = \frac{1}{8}\sqrt{3}$. **[3]**

Cambridge International A Level Mathematics 9709 Paper 31 Q4 June 2010

Example 33

a Find $\int (4 + \tan^2 2x) \, dx$. **[3]**

b Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin\left(x + \frac{1}{6}\pi\right)}{\sin x} \, dx$. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 June 2015

Integration by recognition

Example 1

Find $\int \left(\frac{2x}{x^2+2}\right) dx.$

Example 2

Find $\int \frac{3x^2+5x}{2x^3+5x^2}\, dx.$

Example 3

Find $\int \left(\frac{e^{2x}}{e^{2x}+3}\right) dx.$

Example 4

Find $\int \left(\frac{1+2xe^{x^2}}{x+e^{x^2}}\right) dx.$

Example 5

Find $\int_0^1 \frac{2e^x}{1+e^x}\, dx$

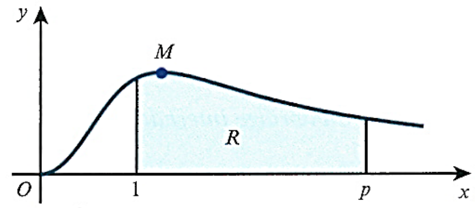
Example 4

Find $\int \tan 2x\, dx.$

Example 5

Find $\int \tan 3x\, dx.$

Example 6



The diagram shows the curve $y = \dfrac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

i Find the exact value of the x-coordinate of M. **[4]**

ii Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. **[6]**

Cambridge International A Level Mathematics 9709 Paper 31 Q10 November 2015

CHAPTER 12: INTEGRATION BY PARTS

Integration by parts

$$\int \left(u\frac{dv}{dx} \right) dx = uv - \int \left(v\frac{du}{dx} \right) dx$$

Example 1

Find $\int x \cos 3x \, dx$.

Example 2

Find $\int x^2 e^{2x} \, dx$.

Example 3

Find $\int x^2 \cos 2x \, dx$.

Example 4

Find $\int (x \ln x) \, dx$.

Example 5

Find $\int (\ln x) \, dx$.

Example 6

Find $\int_1^2 (x^3 \ln x) \, dx$.

Example 7

Find $\int (e^{2x} \sin x) dx$

Example 8

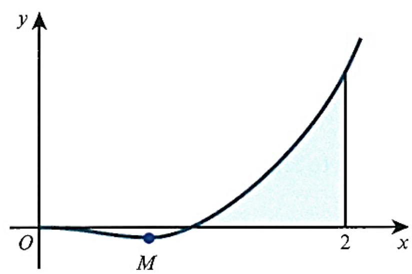
Given that $I = \int \frac{\ln x}{x} \, dx$, use integration by parts to show that $I = \frac{1}{2}(\ln x)^2 + c$.

Example 9

Find the exact value of $\int_0^{\frac{1}{2}} x\mathrm{e}^{-2x}\,\mathrm{d}x$. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q2 June 2016

Example 10



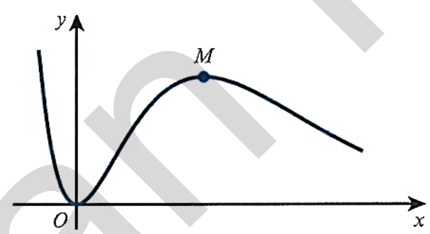
The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

i Find the exact coordinates of M. **[5]**

ii Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 2$. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 November 2010

Example 11



The diagram shows the curve $y = x^2\mathrm{e}^{2-x}$ and its maximum point M.

i Show that the x-coordinate of M is 2. **[3]**

ii Find the exact value of $\int_0^2 x^2\mathrm{e}^{2-x}\,\mathrm{d}x$. **[6]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 June 2015

Example 12

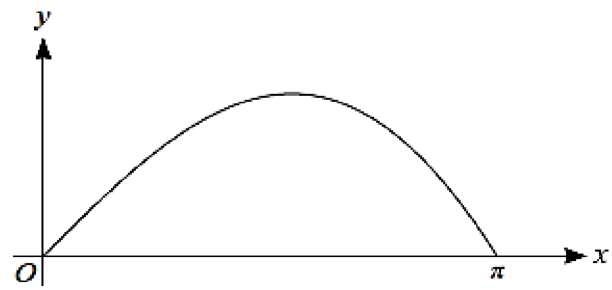
November 2014/31 Question 6

It is given that $\int_1^a \ln(2x)\,\mathrm{d}x = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. **[6]**

Example 13

June 2014/32 Question 8



The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

(i) Find $\frac{dy}{dx}$ and show that $4\frac{d^2y}{dx^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

(ii) Find the exact value of the area of the region enclosed by this part of the curve and the x-axis. [5]

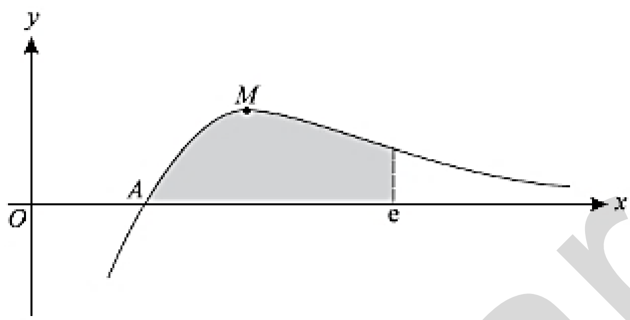
HOMEWORK: INTEGRATIONS BY PARTS VARIANT 32

1 Find the exact value of $\int_0^1 xe^{2x}\,dx$. [4]

J03/Q2

Answer: $\frac{1}{4}(e^2 + 1)$.

2



The diagram shows the curve $y = \frac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = e$. [5]

J04/Q10

Answers: (i) 1; (ii) $x = e^{\frac{1}{2}}$, $y = \frac{2}{e}$; (iii) $1 - \frac{2}{e}$.

4

The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

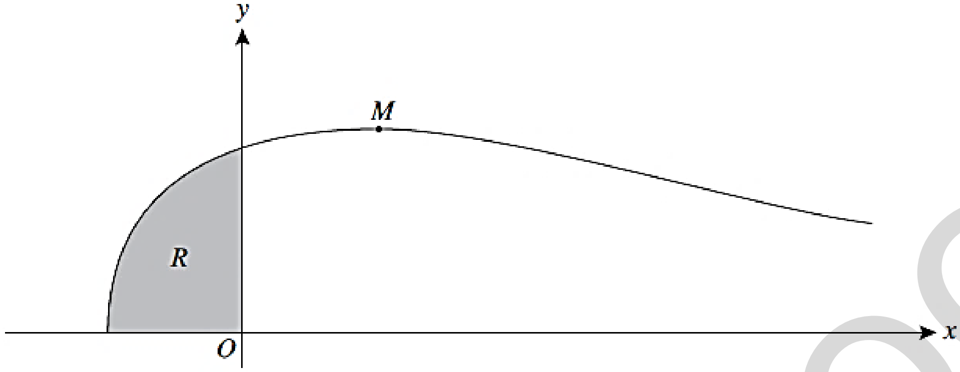
(i) Find the exact value of the x-coordinate of M. [4]

(ii) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

Answers: (i) e^{-2}; (ii) 4.28.

J06/Q8

5



The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

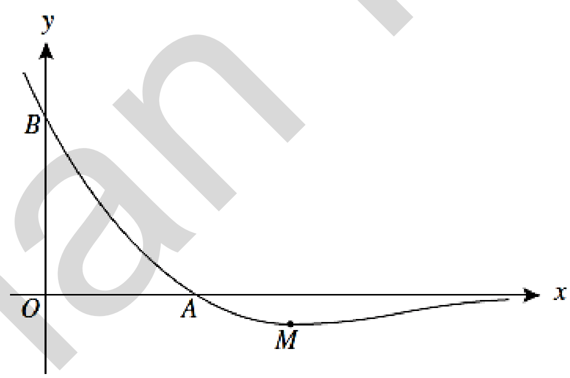
(i) Find the x-coordinate of M. [4]

(ii) Find by integration the volume of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

Answers: (i) $\frac{1}{2}$; (ii) $\pi(2\sqrt{e}-3)$.

J08/Q9

6



The diagram shows the curve $y = (3-x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

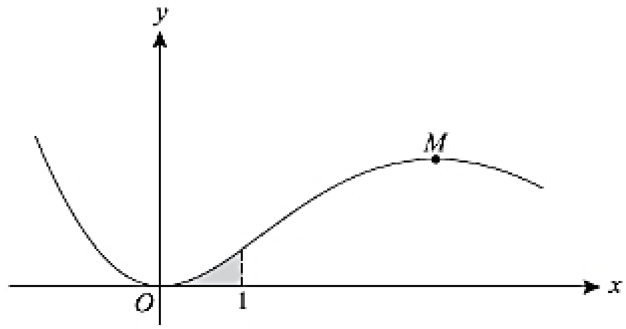
(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of the region bounded by OA, OB and the curve, giving your answer in terms of e. [5]

Answers: (i) $3\frac{1}{2}$; (ii) $\frac{1}{4}(5+e^{-6})$.

N03/Q6

7



The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the x-coordinate of M, the maximum point of the curve. **[4]**

(ii) Find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 1$, giving your answer in terms of e. **[5]**

Answers: (i) 4; (ii) $16 - 26e^{-\frac{1}{2}}$. N04/Q7

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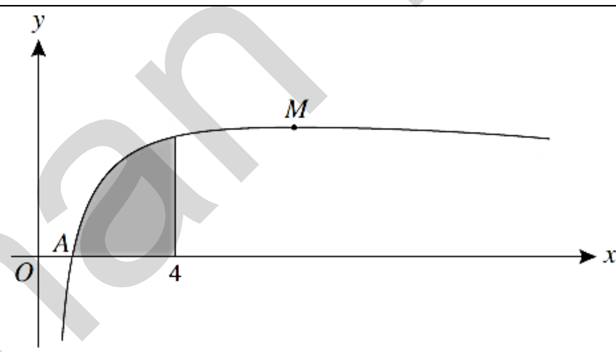
8 Use integration by parts to show that

$$\int_2^4 \ln x \, dx = 6\ln 2 - 2. \qquad [4]$$

N07/Q3

9



The diagram shows the curve $y = \dfrac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.

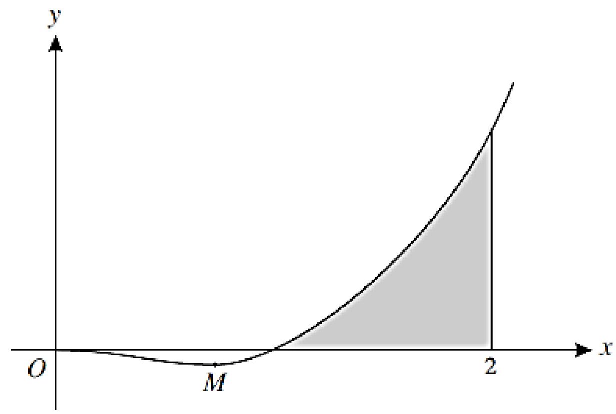
(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

(iii) Using integration by parts, show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 4$ is equal to $8\ln 2 - 4$. [5]

Answers: (i) $(1, 0)$; (ii) e^2. N09/31/Q9

10



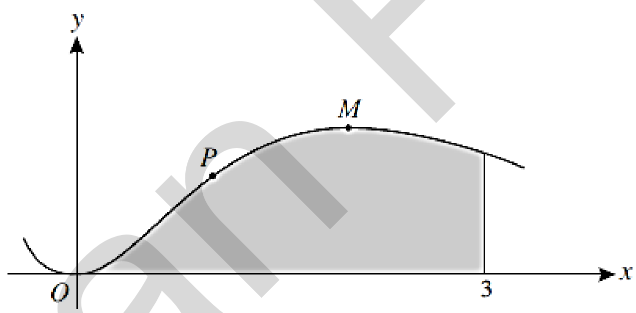
The diagram shows the curve $y = x^3 \ln x$ and its minimum point M.

(i) Find the exact coordinates of M. [5]

(ii) Find the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 2$. [5]

Answers: (i) $(e^{-\frac{1}{3}}, -\frac{1}{3e})$; (ii) $4\ln 2 - \frac{15}{16}$. N10/32/Q9

11



The diagram shows the curve $y = x^2 e^{-x}$.

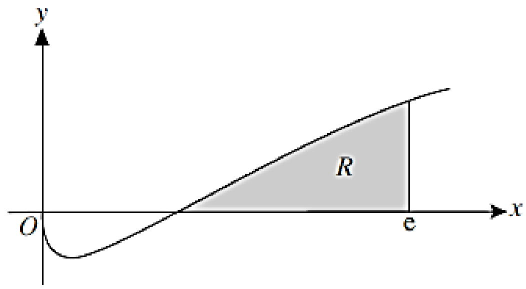
(i) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 3$ is equal to $2 - \frac{17}{e^3}$. [5]

(ii) Find the x-coordinate of the maximum point M on the curve. [4]

(iii) Find the x-coordinate of the point P at which the tangent to the curve passes through the origin. [2]

Answers: **(ii)** 2; **(iii)** 1. J11/32/Q10

12



The diagram shows the curve $y = x^{\frac{1}{2}} \ln x$. The shaded region between the curve, the x-axis and the line $x = \mathrm{e}$ is denoted by R.

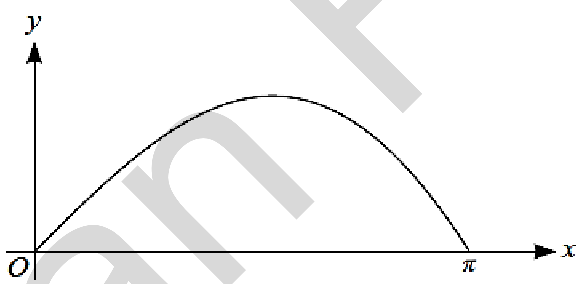
(i) Find the equation of the tangent to the curve at the point where $x = 1$, giving your answer in the form $y = mx + c$. [4]

(ii) Find by integration the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π and e. [7]

Answer. **(i)** $y = x - 1$ **(ii)** $\frac{1}{4}\pi(\mathrm{e}^2 - 1)$

J12/32/Q9

13



The diagram shows the curve $y = x\cos\frac{1}{2}x$ for $0 \leqslant x \leqslant \pi$.

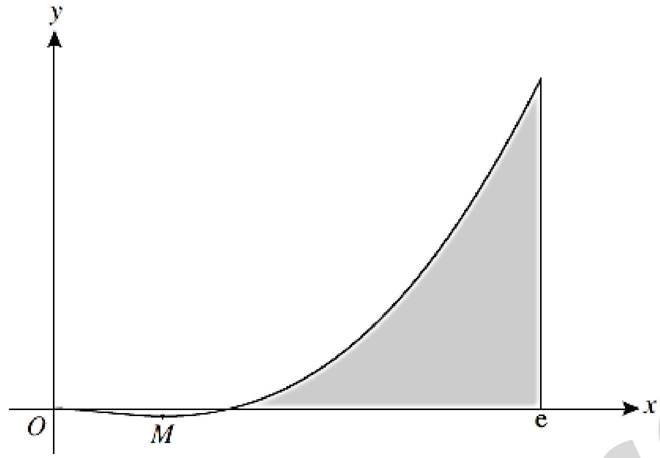
(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and show that $4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y + 4\sin\frac{1}{2}x = 0$. [5]

(ii) Find the exact value of the area of the region enclosed by this part of the curve and the x-axis. [5]

Answer. **(ii)** $2\pi - 4$

J14/32/Q8

14



The diagram shows the curve $y = x^2 \ln x$ and its minimum point M.

(i) Find the exact values of the coordinates of M. [5]

(ii) Find the exact value of the area of the shaded region bounded by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

Answer. **(i)** $x = \frac{1}{\sqrt{\mathrm{e}}}$, $y = -\frac{1}{2\mathrm{e}}$; **(ii)** $\frac{1}{9}(2\mathrm{e}^3 + 1)$ **N11/32/Q9**

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15 Find the exact value of $\int_1^4 \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$. [5]

Answer. 8 ln 2 – 4 **N13/32/Q3**

HOMEWORK: INTEGRATIONS BY PARTS– VARIANTS 31 & 33

1 It is given that $\int_0^a x\cos x\,dx = 0.5$, where $0 < a < \frac{1}{2}\pi$.

(i) Show that a satisfies the equation $\sin a = \dfrac{1.5 - \cos a}{a}$.

Answer:	33/J15/6

2 (ii) Find the exact value of $\int_0^2 x^2 e^{2-x}\,dx$. [6]

Answer: (ii) $2e^2 - 10$	31/J15/9

3 It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \dfrac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

	31/N14/6

4 It is given that $\int_0^p 4xe^{-\frac{1}{2}x}\,dx = 9$, where p is a positive constant.

(i) Show that $p = 2\ln\left(\dfrac{8p+16}{7}\right)$. [5]

Answer:	33/N13/5

5 (a) Show that $\int_2^4 4x\ln x\,dx = 56\ln 2 - 12$. [5]

Answer:	31/J13/8

6 The expression f(x) is defined by $f(x) = 3xe^{-2x}$.

(i) Find the exact value of $f'(-\frac{1}{2})$. [3]

(ii) Find the exact value of $\int_{-\frac{1}{2}}^0 f(x)\,dx$. [5]

Answers: (i) $6e$; (ii) $-\dfrac{3}{4}$.	33/N12/5

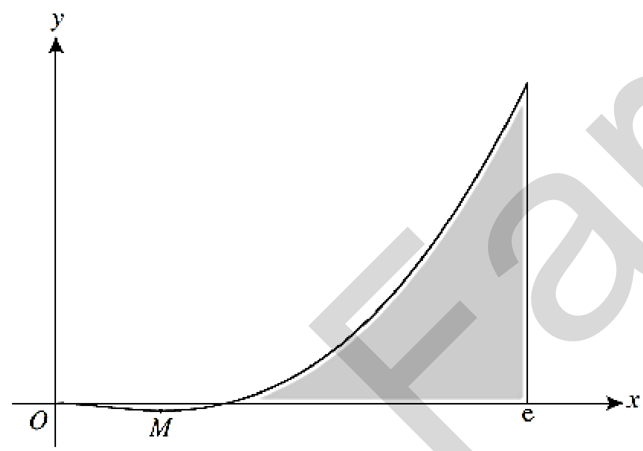
7 It is given that $\int_1^a x \ln x \, dx = 22$, where a is a constant greater than 1.

(i) Show that $a = \sqrt{\left(\frac{87}{2\ln a - 1}\right)}$. [5]

(ii) Use an iterative formula based on the equation in part (i) to find the value of a correct to 2 decimal places. Use an initial value of 6 and give the result of each iteration to 4 decimal places. [3]

Answer: (i) $\sqrt{\frac{87}{2\ln a - 1}}$ (given); (ii) 5.86 33/N11/5

8



The diagram shows the curve $y = x^2 \ln x$ and its minimum point M.

(i) Find the exact values of the coordinates of M. [5]

(ii) Find the exact value of the area of the shaded region bounded by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

Answer: (i) $x = \frac{1}{\sqrt{\mathrm{e}}}, y = -\frac{1}{2\mathrm{e}}$; (ii) $\frac{1}{9}(2\mathrm{e}^3 + 1)$ 31/N11/9

9 Show that $\int_0^1 (1-x)\mathrm{e}^{-\frac{1}{2}x} \, dx = 4\mathrm{e}^{-\frac{1}{2}} - 2$. [5]

Answer: 31/J11/3

10 (i) Given that $\int_1^a \frac{\ln x}{x^2} \, dx = \frac{2}{5}$, show that $a = \frac{5}{3}(1 + \ln a)$. [5]

Answer: 33/N10/7

11 Find the exact value of $\int_0^{\frac{1}{2}} x\mathrm{e}^{-2x} \, dx$. [5]

Answer: $\frac{1}{4} - \frac{1}{2}\mathrm{e}^{-1}$ J16/31/Q2

12 Find the exact value of $\int_0^{\frac{1}{2}\pi} \theta \sin \frac{1}{2}\theta \, d\theta$. [4]

Answer: $(4 - \pi)/\sqrt{2}$ J17/33/Q4

---

13 Showing all necessary working, find the value of $\int_0^{\frac{1}{6}\pi} x \cos 3x \, dx$, giving your answer in terms of π. [5]

Answer: $\frac{(\pi - 2)}{18}$ J18/33/Q3

CHAPTER 13: INTEGRATION BY SUSBTITUTION

Example 1

Find $\int \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

Example 2

Find $\int_1^2 x\sqrt{5x-2}\,dx$ using the substitution $u = 5x - 2$. Give your answer to 3 significant figures.

Example 3

Find $\int_1^e \left(\frac{1}{x}\ln x\right)dx$ using the substitution $x = e^u$.

Example 4

Find $\int \frac{1}{(5x-2)^2}\,dx$ using the substitution $u = 5x - 2$.

Example 5

Find $\int_0^{\frac{\pi}{4}} \cos 2x \sin^3 2x\,dx$ using the substitution $u = \sin 2x$.

Example 6

Find $\int_0^{\frac{1}{12}\pi} \left(\frac{\cos 2x}{1+\sin 2x}\right)dx$ using the substitution $u = 1 + \sin 2x$.

Example 7

Use the substitution $u = 2x+1$ to find $\int 4x\sqrt{2x+1}\,dx$.

Example 8

Use the substitution $x = 2\tan u$ to find $\int \frac{8}{x^2+4}\,dx$.

Example 9

Use the substitution $u = \sin x$ to find $\int \sin^2 2x \cos x\,dx$.

Example 10

Use the substitution $u = x+1$ to find $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$.

Example 11

Use the substitution $u = 1 + 3\tan x$ to find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1+3\tan x)}}{\cos^2 x}\,dx$ **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q2 June 2014

Example 12

Let $I = \displaystyle\int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\,dx$.

i Using the substitution $x = 2\sin\theta$, show that $I = \displaystyle\int_0^{\frac{1}{6}\pi} 4\sin^2\theta\,d\theta$. **[3]**

ii Hence find the exact value of I. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 November 2010

Example 13

The integral I is defined by $I = \displaystyle\int_0^2 4t^3 \ln(t^2+1)\,dt$.

i Use the substitution $x = t^2 + 1$ to show that $I = \displaystyle\int_1^5 (2x-2)\ln x\,dx$. **[3]**

ii Hence find the exact value of I. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q7 June 2011

Example 14

i Show that $\displaystyle\int_2^4 4x\ln x\,dx = 56\ln 2 - 12$. **[5]**

ii Use the substitution $u = \sin 4x$ to find the exact value of $\displaystyle\int_0^{\frac{1}{24}\pi} \cos^3 4x\,dx$. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q8 June 2013

Example 15

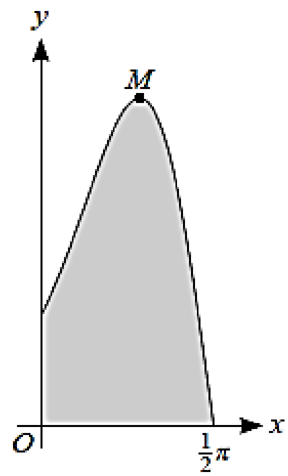
June 2014/32 Question 2

Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1+3\tan x)}}{\cos^2 x}\,dx.$$ [5]

Example 16

June 2014/33 Question 9



The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. [5]

HOMEWORK: INTEGRATION BY SUBSTITUTION
VARIANT 32

1 (i) Prove the identity

$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x.$$ [3]

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2}\ln 2$. [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$, giving your answer in the form $a \ln b$. [4]

J03/Q10

Answer: **(iii)** $\frac{1}{4}\ln 3$.

---

2 (i) Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1 - \cos 4\theta).$$ [3]

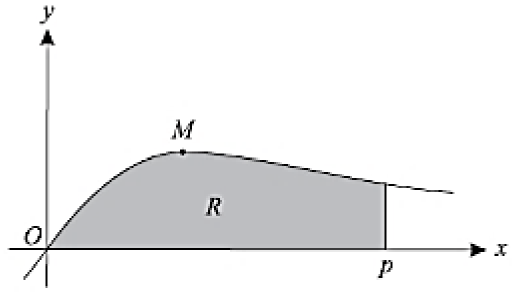
(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta \, d\theta.$$ [3]

J04/Q5

Answer: **(ii)** $\frac{1}{24}\pi + \frac{\sqrt{3}}{64}$.

3



The diagram shows part of the curve $y = \dfrac{x}{x^2+1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of R in terms of p. [3]

(iii) Hence calculate the value of p for which the area of R is 1, giving your answer correct to 3 significant figures. [2]

Answers: (i) 1; (ii) $\dfrac{1}{2}\ln(p^2+1)$; (iii) 2.53. J05/Q9

4 (i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{\left(\cos\theta + (\sqrt{3})\sin\theta\right)^2}\,d\theta = \frac{1}{\sqrt{3}}$. [4]

Answer: (i) $2\cos\left(\theta - \dfrac{1}{3}\pi\right)$. J07/Q5

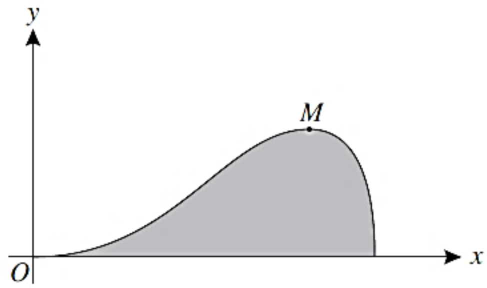
5 Let $I = \displaystyle\int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \displaystyle\int_1^2 \frac{2}{u(4-u)}\,du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

J07/Q7

6



The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Show, by means of the substitution $x = \sin\theta$, that the area A of the shaded region between the curve and the x-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta \, d\theta.$$ [3]

(iii) Hence obtain the exact value of A. [4]

Answers: (i) $\frac{\sqrt{6}}{3}$; (iii) $\frac{1}{16}\pi$. J09/Q10

7 Show that $\int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4$. [5]

J10/32/Q2

8 Find the exact value of $\int_1^2 x \ln x \, dx$. [4]

Answer: $2\ln 2 - \frac{3}{4}$. N02/Q2

9 (i) Use the substitution $x = \sin^2\theta$ to show that

$$\int \sqrt{\left(\frac{x}{1-x}\right)} \, dx = \int 2\sin^2\theta \, d\theta.$$ [4]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)} \, dx.$$ [4]

Answer: (ii) $\frac{1}{6}\pi - \frac{1}{4}\sqrt{3}$. N05/Q6

10 **(i)** Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. [4]

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta \, d\theta.$$ [4]

Answer: **(ii)** $\frac{1}{32}(2\pi - \sqrt{3})$. N09/31/Q5

11 Let $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}} \, dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta \, d\theta.$$ [3]

(ii) Hence find the exact value of I. [4]

Answer: **(ii)** $\frac{1}{3}\pi - \frac{\sqrt{3}}{2}$. N10/32/Q5

12 **(i)** Use the substitution $x = \tan\theta$ to show that

$$\int \frac{1-x^2}{(1+x^2)^2} \, dx = \int \cos 2\theta \, d\theta.$$ [4]

(ii) Hence find the value of

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2} \, dx.$$ [3]

Answer: **(ii)** $\frac{1}{2}$. J05/Q4

13 **(i)** Use the substitution $x = 2\tan\theta$ to show that

$$\int_0^2 \frac{8}{(4+x^2)^2} \, dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta \, d\theta.$$ [4]

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4+x^2)^2} \, dx.$$ [4]

Answer: **(ii)** $\frac{1}{8}\pi + \frac{1}{4}$. N09/32/Q6

14 Let $I = \int_2^5 \frac{5}{x + \sqrt{(6 - x)}}\, dx$.

(i) Using the substitution $u = \sqrt{(6 - x)}$, show that

$$I = \int_1^2 \frac{10u}{(3 - u)(2 + u)}\, du. \qquad [4]$$

(ii) Hence show that $I = 2\ln(\frac{9}{2})$. [6]

J12/32/Q8

---

15 (i) By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3}) \tan \theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3 + x^2)}}\, dx,$$

expressing your answer as a single logarithm. [4]

Answer: $\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$

J13/32/Q6

---

16 Let $I = \int_0^1 \frac{\sqrt{x}}{2 - \sqrt{x}}\, dx$.

(i) Using the substitution $u = 2 - \sqrt{x}$, show that $I = \int_1^2 \frac{2(2 - u)^2}{u}\, du$. [4]

(ii) Hence show that $I = 8\ln 2 - 5$. [4]

J15/32/Q6

17 **(i)** By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. [2]

(ii) Show that $\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}(8\sqrt{2} - \pi)$. [3]

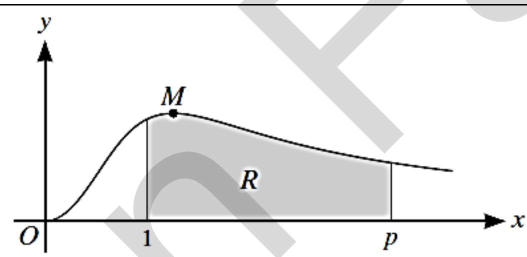
N12/32/Q5

18 **(i)** Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2}\ln 3$. [4]

N13/32/Q5

19



The diagram shows the curve $y = \frac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

Answers: (i) $\sqrt[3]{2}$; (ii) 3.40

N15/32/Q10

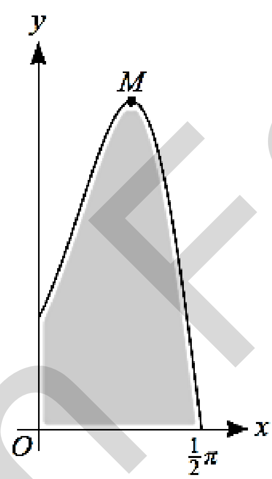
HOMEWORK: INTEGRATION BY SUBSTITUTION– VARIANTS 31 & 33

1 (a) Find $\int (4 + \tan^2 2x)\,dx$. [3]

(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin\left(x + \frac{1}{6}\pi\right)}{\sin x}\,dx$. [5]

Answers: **(i)** $3x + \frac{1}{2}\tan 2x + c$ **(ii)** $\frac{1}{8}\pi\sqrt{3} - \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\right)$ 31/J15/5

2



The diagram shows the curve $y = e^{2\sin x}\cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Using the substitution $u = \sin x$, find the exact value of the area of the shaded region bounded by the curve and the axes. [5]

(ii) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [6]

Answers: **(i)** $\frac{1}{2}(e^2 - 1)$ **(ii)** 0.896 33/J14/9

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3 Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\,dx. \quad [5]$$

Answer: $\frac{14}{9}$ 31/J14/2

4 Use the substitution $u = 3x + 1$ to find $\displaystyle\int \frac{3x}{3x+1}\,dx$. [4]

Answer: $\frac{1}{3}(3x+1) - \frac{1}{3}\ln|3x+1| + c$ 33/N13/2

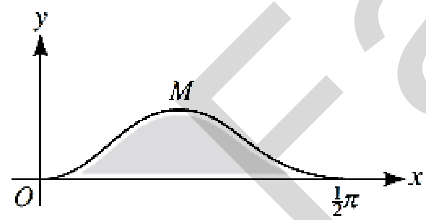
5 (i) Express $(\sqrt{3})\cos x + \sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{1}{\left((\sqrt{3})\cos x + \sin x\right)^2}\,dx = \tfrac{1}{4}\sqrt{3}.$$ [4]

Answer: 2 and $\frac{\pi}{6}$ 33/J13/4

6



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [6]

(ii) Using the substitution $u = \sin x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [4]

Answer: 0.685 Answer: $\frac{8}{15}$ 33/J13/9

7 (b) Use the substitution $u = \sin 4x$ to find the exact value of $\displaystyle\int_0^{\frac{1}{24}\pi} \cos^3 4x\,dx$. [5]

Answer: $\frac{11}{96}$ 31/J13/8

8 (i) Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. [3]

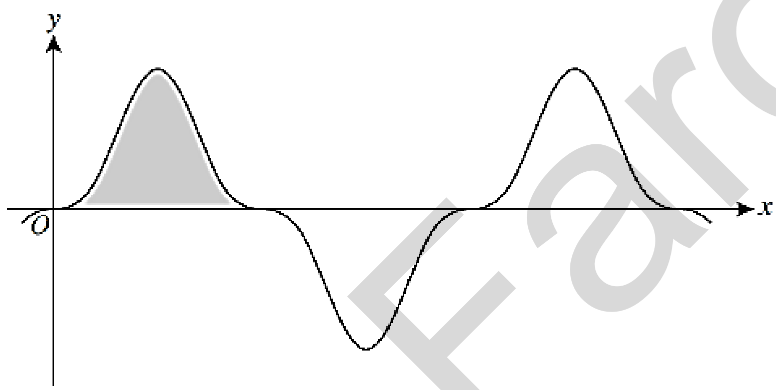
(ii) Hence

(a) solve the equation $4\cos\theta + 3\sin\theta = 2$ for $0 < \theta < 2\pi$, [4]

(b) find $\displaystyle\int \frac{50}{(4\cos\theta + 3\sin\theta)^2}\,d\theta$. [3]

Answer: 0.6435 *Answer*: 1.80 and 5.77 *Answer*: 2tan(θ – 0.6435) 31/J13/9

9



The diagram shows part of the curve $y = \sin^3 2x \cos^3 2x$. The shaded region shown is bounded by the curve and the x-axis and its exact area is denoted by A.

(i) Use the substitution $u = \sin 2x$ in a suitable integral to find the value of A. [6]

(ii) Given that $\displaystyle\int_0^{k\pi} |\sin^3 2x \cos^3 2x|\,dx = 40A$, find the value of the constant k. [2]

Answers: (i) $\frac{1}{24}$; (ii) 10. 33/N12/7

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10 (i) By differentiating $\frac{1}{\cos x}$, show that if $y = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$. [2]

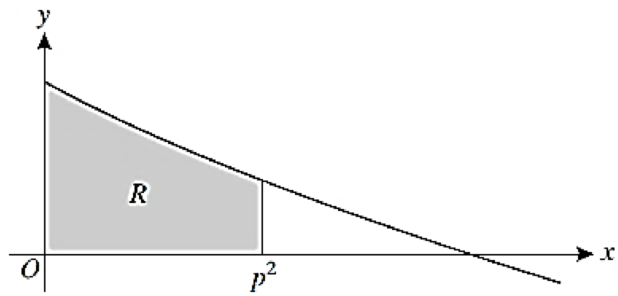
(ii) Show that $\displaystyle\frac{1}{\sec x - \tan x} \equiv \sec x + \tan x$. [1]

(iii) Deduce that $\displaystyle\frac{1}{(\sec x - \tan x)^2} \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1}{(\sec x - \tan x)^2}\,dx = \frac{1}{4}(8\sqrt{2} - \pi)$. [3]

Answer: 31/N12/5

11

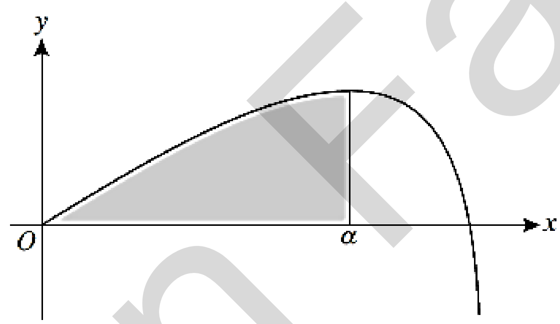


The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where x is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by R. The area of R is equal to 1.

(i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \dfrac{3 - 2\cos p}{2p}$. [6]

Answer: 33/J12/7

12



The diagram shows the curve

$$y = 8\sin\tfrac{1}{2}x - \tan\tfrac{1}{2}x$$

for $0 \leqslant x < \pi$. The x-coordinate of the maximum point is α and the shaded region is enclosed by the curve and the lines $x = \alpha$ and $y = 0$.

(i) Show that $\alpha = \frac{2}{3}\pi$. [3]

(ii) Find the exact value of the area of the shaded region. [4]

31/J12/5

Answer: (ii) $8 + 2\ln\dfrac{1}{2}$

13 **(i)** Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x)\,dx = \frac{1}{n+1}.$$ [4]

(ii) Hence find the exact value of

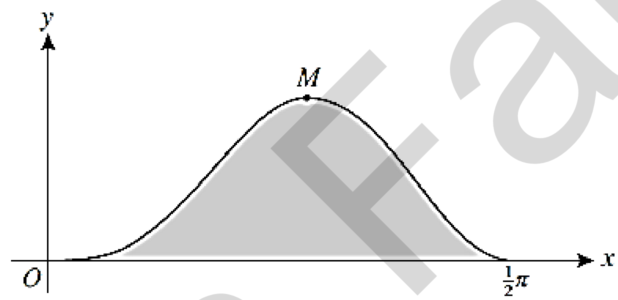
(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x)\,dx$, [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x)\,dx$. [3]

33/N11/10

Answers: (i) $\frac{1}{n+1}$ (given); (ii) (a) $\frac{1}{3}$, (b) $\frac{25}{24}$

14



The diagram shows the curve $y = 5\sin^3 x \cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(i) Find the x-coordinate of M. [5]

(ii) Using the substitution $u = \cos x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [5]

33/J11/8

Answers: (i) 0.886; (ii) $\frac{2}{3}$.

15

The integral I is defined by $I = \int_0^2 4t^3 \ln(t^2 + 1)\,dt$.

(i) Use the substitution $x = t^2 + 1$ to show that $I = \int_1^5 (2x - 2)\ln x\,dx$. [3]

(ii) Hence find the exact value of I. [5]

31/J11/7

Answer: (ii) $15\ln 5 - 4$.

16 (i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, [3]

(b) find the exact value of $\int_0^{\frac{1}{4}\pi} \cos^4\theta \, d\theta$. [3]

Answers: **(ii)(a)** ± 0.572, **(b)** $\frac{3}{32}\pi + \frac{1}{4}$. 31/J11/9

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17 It is given that $f(x) = 4\cos^2 3x$.

(i) Find the exact value of $f'(\frac{1}{9}\pi)$. [3]

(ii) Find $\int f(x)\, dx$. [3]

Answers: **(i)** $-6\sqrt{3}$; **(ii)** $2x + \frac{1}{3}\sin 6x + c$. 33/N10/4

18

Let $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\, dx$.

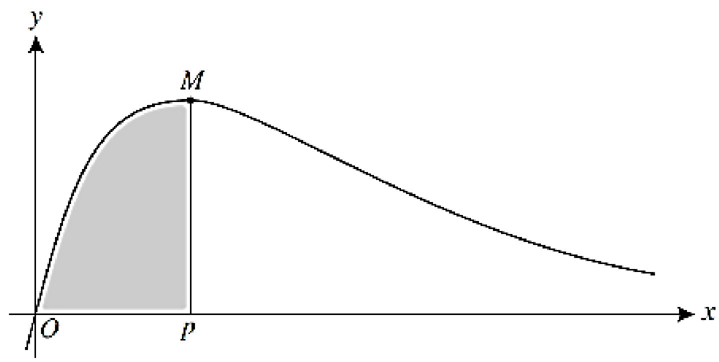
(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta \, d\theta.$$ [3]

(ii) Hence find the exact value of I. [4]

Answer: **(ii)** $\frac{1}{3}\pi - \frac{\sqrt{3}}{2}$. 31/N10/5

19



The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. [4]

(ii) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = p$ is equal to $\frac{1}{8}$. [4]

Answer: (i) ln2. 33/J10/5

20 **(i)** Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta \, d\theta.$$ [4]

33/J10/7

Answer: (ii) $\frac{2}{3} - \frac{3}{8}\sqrt{3}$.

21 **(i)** Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

$$\frac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x \, dx = \frac{1}{8}\sqrt{3}.$$ [3]

22 Use the substitution $u = 4 - 3\cos x$ to find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{9\sin 2x}{\sqrt{(4 - 3\cos x)}} \, dx$. [8]

Answer: $\frac{20}{3}$ N15/33/Q5

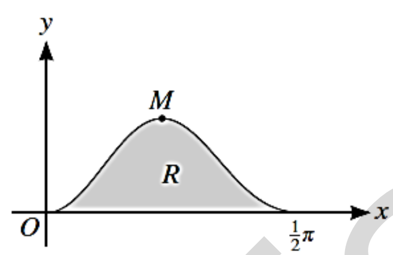
23 Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1}\,du$. [3]

(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

N16/33/Q6

24



The diagram shows the curve $y = 5\sin^2 x\cos^3 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M. The shaded region R is bounded by the curve and the x-axis.

(i) Find the x-coordinate of M, giving your answer correct to 3 decimal places. [5]

(ii) Using the substitution $u = \sin x$ and showing all necessary working, find the exact area of R. [4]

Answers: **(i)** $x = 0.685$ **(ii)** $\frac{2}{3}$

N18/33/Q7

25 Let $I = \int_0^1 \frac{x^5}{(1+x^2)^3}\,dx$.

(i) Using the substitution $u = 1 + x^2$, show that $I = \int_1^2 \frac{(u-1)^2}{2u^3}\,du$. [3]

(ii) Hence find the exact value of I. [5]

Answer: $\frac{1}{2}\ln 2 - \frac{5}{16}$

J16/33/Q7

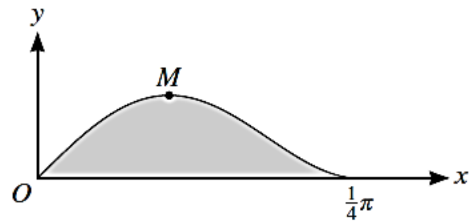
26 It is given that $x = \ln(1-y) - \ln y$, where $0 < y < 1$.

(i) Show that $y = \frac{e^{-x}}{1+e^{-x}}$. [2]

(ii) Hence show that $\int_0^1 y\,dx = \ln\left(\frac{2e}{e+1}\right)$. [4]

J17/31/Q3

27



The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$ and its maximum point M.

(i) Using the substitution $u = \cos x$, find by integration the exact area of the shaded region bounded by the curve and the x-axis. [6]

(ii) Find the x-coordinate of M. Give your answer correct to 2 decimal places. [6]

Answers: **(i)** $\frac{1}{15}\left(7-4\sqrt{2}\right)$ **(ii)** 0.32

J17/31/Q10

28

Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,d\theta$. [4]

(ii) Hence find the exact value of I. [4]

Answer: **(ii)** $\frac{\pi}{6}$

J18/31/Q5

29

Throughout this question the use of a calculator is not permitted.

(i) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact values of R and $\tan\alpha$. [3]

(ii) Hence, showing all necessary working, show that $\int_0^{\frac{1}{4}\pi} \frac{15}{(\cos\theta + 2\sin\theta)^2}\,d\theta = 5$. [5]

Answer: **(i)** $R = \sqrt{5}$, $\tan\alpha = 2$

J18/33/Q7

CHAPTER 14: INTEGRATION WITH PARTIAL FRACTIONS

Example 1

Find $\displaystyle\int \frac{x+3}{(x-2)(x+1)}\,dx.$

Example 2

Find $\displaystyle\int \frac{3x^2+2}{x(x-1)^2}\,dx.$

Example 3

Find $\displaystyle\int \frac{2x^2-x+6}{(x+1)(x^2+2)}\,dx.$

Example 4

By first expressing $\dfrac{4}{x^2(x-2)}$ in partial fractions, find $\displaystyle\int\left(\frac{4}{x^2(x-2)}\right)dx.$

Example 5

By first expressing $\dfrac{2x-4}{x(x^2+4)}$ in partial fractions, find $\displaystyle\int\left(\frac{2x-4}{x(x^2+4)}\right)dx.$

Example 6

Find $\displaystyle\int \frac{3-x}{(x+1)(x^2+3)}\,dx.$

Example 7

Find $\displaystyle\int\left(\frac{3x^3-4x^2-2}{x^2(x-1)}\right)dx.$

Example 8

Evaluate $\displaystyle\int_4^6 \frac{1}{(x-2)(x-3)}\,dx.$

Example 9

June 2014/33 Question 8

Let $f(x) = \dfrac{6+6x}{(2-x)(2+x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{2+x^2}$. [4]

(ii) Show that $\displaystyle\int_{-1}^{1} f(x)\,dx = 3\ln 3$. [5]

Example 10

i Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. **[2]**

ii Using your answer to **part i**, show that $\left(\dfrac{2}{(x+1)(x+3)}\right)^2 \equiv \dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} + \dfrac{1}{x+3} + \dfrac{1}{(x+3)^2}$. **[2]**

iii Hence show that $\displaystyle\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx = \frac{7}{12} - \ln\frac{3}{2}$. **[5]**

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Example 11

By first expressing $\dfrac{4x^2+5x+3}{2x^2+5x+2}$ in partial fractions, show that $\displaystyle\int_0^4 \frac{4x^2+5x+3}{2x^2+5x+2}\,dx = 8 - \ln 9$. **[10]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 June 2012

Example 12

Let $f(x) = \dfrac{12+8x-x^2}{(2-x)(4+x^2)}$.

i Express $f(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{4+x^2}$. **[4]**

ii Show that $\displaystyle\int_0^1 f(x)\,dx = \ln\left(\frac{25}{2}\right)$. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q8 November 2011

HOMEWORK: INTEGRATION BY PARTIAL FRACTIONS VARIATN32

1 An appropriate form for expressing $\dfrac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1} + \frac{B}{x-2},$$

where A and B are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\dfrac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$. [2]

(b) Show that $\displaystyle\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx = \ln 5.$ [6]

N04/Q8

Answer: (a)(i) $\dfrac{A}{x+4}+\dfrac{Bx+C}{x^2+3}$; (ii) $\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$ or $\dfrac{A}{x-2}+\dfrac{Bx+C}{(x+2)^2}$.

---

2 Find the exact value of the constant k for which $\displaystyle\int_1^k \frac{1}{2x-1}\,dx = 1.$ [4]

N07/Q1

Answer: $\frac{1}{2}(e^2+1)$.

---

3 (i) Find the values of the constants A, B, C and D such that

$$\frac{2x^3-1}{x^2(2x-1)} \equiv A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{2x-1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3-1}{x^2(2x-1)}\,dx = \tfrac{3}{2} + \tfrac{1}{2}\ln\left(\tfrac{16}{27}\right).$$ [5]

J10/Q10

Answers: (i) 1, 2, 1, –3.

---

4 Let $f(x) = \dfrac{12+8x-x^2}{(2-x)(4+x^2)}$.

(i) Express $f(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{4+x^2}$. [4]

(ii) Show that $\displaystyle\int_0^1 f(x)\,dx = \ln\left(\tfrac{25}{2}\right)$. [5]

Answer: **(i)** $\dfrac{3}{2-x}+\dfrac{4x}{4+x^2}$ N11/32/Q8

HOMEWORK: INTEGRATION BY PARTIAL FRACTIONS– VARIANTS 31 & 33

1 Let $\mathrm{f}(x) = \dfrac{11x+7}{(2x-1)(x+2)^2}$.

(i) Express $\mathrm{f}(x)$ in partial fractions.

(ii) Show that $\displaystyle\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = \tfrac{1}{4} + \ln\left(\tfrac{9}{4}\right)$.

33/J15/10

Answers: (i) $\dfrac{2}{2x-1} + \dfrac{-1}{x+2} + \dfrac{3}{(x+2)^2}$

2 By first using the substitution $u = \mathrm{e}^x$, show that

$$\int_0^{\ln 4} \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} + 3\mathrm{e}^x + 2}\,\mathrm{d}x = \ln\left(\tfrac{8}{5}\right).$$

Answer: 33/N14/10

3 Let $\mathrm{f}(x) = \dfrac{6+6x}{(2-x)(2+x^2)}$.

(i) Express $\mathrm{f}(x)$ in the form $\dfrac{A}{2-x} + \dfrac{Bx+C}{2+x^2}$.

(ii) Show that $\displaystyle\int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x = 3\ln 3$.

Answer: (i) $A = 3$, $B = 3$, $C = 0$ 33/J14/8

4 Express $\dfrac{7x^2-3x+2}{x(x^2+1)}$ in partial fractions.

Answer: $\dfrac{2}{x} + \dfrac{5x-3}{x^2+1}$ 31/J13/3

5 Let $\mathrm{f}(x) = \dfrac{4x^2-7x-1}{(x+1)(2x-3)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5

(ii) Show that $\displaystyle\int_2^6 \mathrm{f}(x)\,\mathrm{d}x = 8 - \ln\left(\tfrac{49}{3}\right)$. [5

33/J12/8

Answer: (ii) $2 - \dfrac{2}{x+1} - \dfrac{1}{2x-3}$

6 By first expressing $\frac{4x^2+5x+3}{2x^2+5x+2}$ in partial fractions, show that

$$\int_0^4 \frac{4x^2+5x+3}{2x^2+5x+2}\,dx = 8 - \ln 9.$$

Answer: 31/J12/9

7 Let $f(x) = \frac{12+8x-x^2}{(2-x)(4+x^2)}$.

(i) Express $f(x)$ in the form $\frac{A}{2-x} + \frac{Bx+C}{4+x^2}$.

(ii) Show that $\int_0^1 f(x)\,dx = \ln\left(\frac{25}{2}\right)$.

Answer: (i) $\frac{3}{2-x} + \frac{4x}{4+x^2}$ 31/N11/8

8 Show that $\int_0^7 \frac{2x+7}{(2x+1)(x+2)}\,dx = \ln 50$.

Answer: 33/N10/5

9 (i) Express $\frac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part (i), show that

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 \equiv \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}.$$ [2]

(iii) Hence show that $\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx = \frac{7}{12} - \ln\frac{3}{2}$. [5]

Answer: (i) $\frac{1}{x+1} - \frac{1}{x+3}$. 31/J10/8

10 (i) Show that $(x+1)$ is a factor of $4x^3 - x^2 - 11x - 6$. [2]

(ii) Find $\int \frac{4x^2+9x-1}{4x^3-x^2-11x-6}\,dx$. [8]

Answer: $-2\ln(x+1) + \ln(x-2) + 2\ln(4x+3) + C$ N15/33/Q7

11 Let $f(x) = \dfrac{6x^2 + 8x + 9}{(2 - x)(3 + 2x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence, showing all necessary working, show that $\int_{-1}^{0} f(x)\,dx = 1 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$. [5]

Answers: **(i)** $\dfrac{1}{(2-x)} - \dfrac{1}{(3+2x)} + \dfrac{3}{(3+2x)^2}$ or $\dfrac{1}{(2-x)} - \dfrac{2x}{(3+2x)^2}$ **N18/33/Q9**

---

12 Let $f(x) = \dfrac{3x^2 - 4}{x^2(3x + 2)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_{1}^{2} f(x)\,dx = \ln\left(\frac{25}{8}\right) - 1$. [5]

Answer: **(i)** $\dfrac{3}{x} - \dfrac{2}{x^2} - \dfrac{6}{3x+2}$ **J17/33/Q9**

CHAPTER 15: DIFFERENTIATION AND INTEGRATION OF $tan^{-1}(x)$

Proof

It is given that $y = \tan^{-1} x$. Find an expression for $\frac{dy}{dx}$.

If $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$.

If $y = \tan^{-1} ax$, then, using the chain rule with $u = ax$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+(ax)^2} \times a$$

Example 1

If $y = \tan^{-1}(1-2x)$, find, in its simplest form, an expression for $\frac{dy}{dx}$.

Example 2

Differentiate with respect to x.

a $\tan^{-1} 3x$

b $\tan^{-1} \sqrt{x}$

c $\tan^{-1}\left(\frac{x}{x-2}\right)$

Example 3

$y = x \tan^{-1} x$

Example 4

$y = \frac{1}{4} \tan^{-1} 2x$

Integration of $\frac{1}{x^2+a^2}$

$$\boxed{\int\left(\frac{1}{1+x^2}\right)\mathrm{d}x = \tan^{-1}x + c}$$

Example 5

$y = \tan^{-1}\left(\frac{x}{a}\right)$. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example 6

$y = \tan^{-1}(bx)$. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\boxed{\int\left(\frac{1}{a^2+x^2}\right)\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

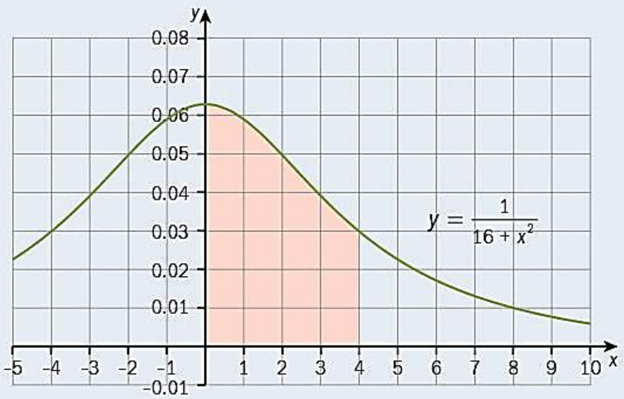
$$\boxed{\int\left(\frac{1}{1+b^2x^2}\right)\mathrm{d}x = \frac{1}{b}\tan^{-1}(bx) + c}$$

Example 7

Find $\int\left(\frac{1}{x^2+25}\right)\mathrm{d}x$.

Example 8

The diagram shows the graph of $y = \dfrac{1}{16 + x^2}$.



Calculate the exact value of the shaded area.

Example 9

Find $\displaystyle\int \frac{1}{1+9x^2}\,dx$.

Example 10

Find $\displaystyle\int \frac{1}{2x^2+3}\,dx$.

Example 11

Find the exact value of $\displaystyle\int_0^2 \frac{1}{x^2+4}\,dx$.

Example 12

Find $\displaystyle\int \frac{1}{16+25x^2}\,dx$.

Example 13

2020 Specimen Paper 3 Question 5

(a) Show that $\dfrac{d}{dx}(x - \tan^{-1}x) = \dfrac{x^2}{1+x^2}$. [2]

(b) Show that $\displaystyle\int_0^{\sqrt{3}} x\tan^{-1}x\,dx = \tfrac{2}{3}\pi - \tfrac{1}{2}\sqrt{3}$. [5]

Homework Exercise

Find the following integrals.

a) $\int \frac{1}{4+9x^2}\,dx$ b) $\int \frac{12}{9+16x^2}\,dx$ c) $\int \frac{1}{49x^2+81}\,dx$

Calculate the exact value of the following definite integrals.

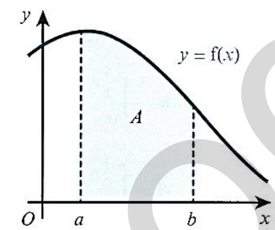
a) $\int_{-3}^{3} \frac{1}{9+x^2}\,dx$ b) $\int_{\sqrt{3}}^{3} \frac{\sqrt{3}}{3+x^2}\,dx$ c) $\int_{0}^{5} \frac{20}{25+x^2}\,dx$

Calculate the exact value of the following definite integrals.

a) $\int_{0}^{\frac{\sqrt{3}}{2}} \frac{1}{1+4x^2}\,dx$ b) $\int_{-0.2}^{0.2} \frac{1}{1+25x^2}\,dx$ c) $\int_{\frac{1}{4\sqrt{3}}}^{\frac{\sqrt{3}}{4}} \left(\frac{8}{1+16x^2}\right)dx$

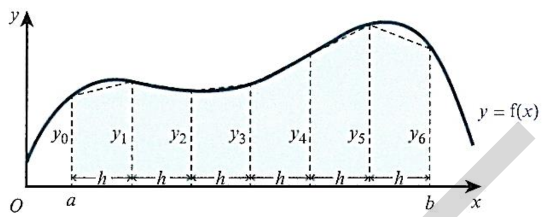
CHAPTER 16: TRAPEZIUM RULE

We already know that the area, A, under the curve $y = f(x)$ between $x = a$ and $x = b$ can be found by evaluating $\int_a^b f(x)\,dx$. Sometimes we might not be able to find the value of $\int_a^b f(x)\,dx$ by the integration methods that we have covered so far or maybe the function cannot be integrated algebraically.



In these situations an approximate answer can be found using a numerical method called the **trapezium rule**.

This numerical method involves splitting the area under the curve $y = f(x)$ between $x = a$ and $x = b$ into equal width strips.



The area of each trapezium-shaped strip can be found using the formula:

$$\text{area} = \frac{1}{2}(\text{sum of parallel sides}) \times \text{width}$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})] \quad \text{where } h = \frac{b-a}{n}.$$

An easy way to remember this rule in terms of the ordinates is:

$$\int_a^b f(x)\,dx \approx \text{half width of strip} \times (\text{first} + \text{last} + \text{twice the sum of all the others}).$$

Example 1

The diagram shows the curve $y = x - 2\ln x$.

Use the trapezium rule with 2 intervals to estimate the shaded area, giving your answer correct to 2 decimal places. State, with a reason, whether the trapezium rule gives an under-estimate or an over-estimate of the true value.

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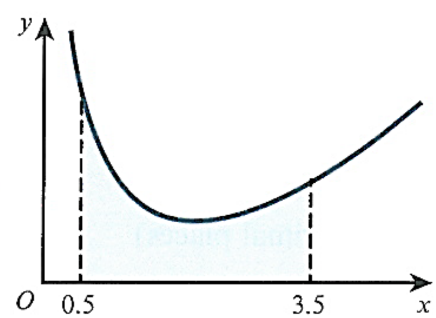
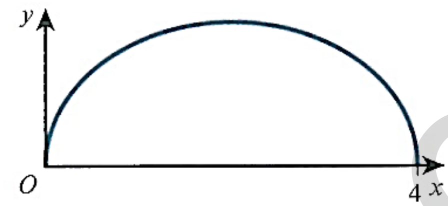


Example 2

The diagram shows the curve $y = 2\sqrt{4x - x^2}$.



Use the trapezium rule with 4 intervals to estimate the value of $\int_0^4 2\sqrt{4x - x^2}\,dx$ giving your answer correct to 2 decimal places. State, with a reason, whether the trapezium rule gives an under-estimate or an over-estimate of the true value.

Example 3

a) Use the trapezium rule with five intervals to estimate $\int_0^{\frac{\pi}{2}} (x\cos x)\,dx$, giving your answer correct to 3 decimal places.

b) In Example 21, you saw that the exact value of this integral is $\frac{\pi}{2} - 1$. Calculate the percentage error in the estimate using the trapezium rule with five intervals.

Example 4

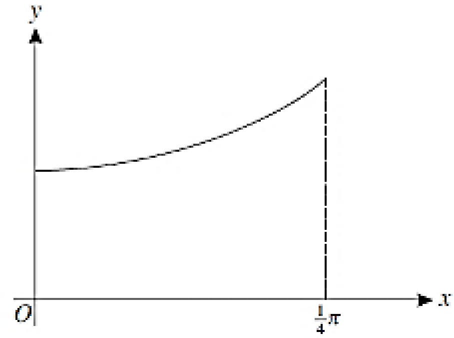
i Use the trapezium rule with two intervals to estimate the value of $\int_0^1 \frac{1}{6 + 2e^x}\,dx$, giving your answer correct to 2 decimal places. **[3]**

ii Find $\int \frac{(e^x - 2)^2}{e^{2x}}\,dx$. **[4]**

Cambridge International AS & A Level Mathematics 9709 Paper 21 Q6 November 2012

HOMEWORK: TRAPEZIUM RULE VARIANT 32

1



The diagram shows the curve $y = \sqrt{(1 + 2\tan^2 x)}$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$.

(i) Use the trapezium rule with three intervals to estimate the value of

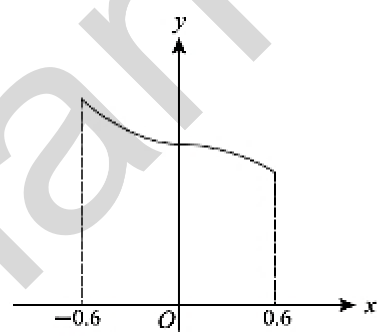
$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 + 2\tan^2 x)}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) The estimate found in part (i) is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than E or less than E. [1]

Answer. (i) 0.98. J09/Q2

2



The diagram shows a sketch of the curve $y = \frac{1}{1 + x^3}$ for values of x from -0.6 to 0.6.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-0.6}^{0.6} \frac{1}{1 + x^3}\,dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

Answer (i) 1.23 J05/Q2

3 Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{2}\pi} \ln(1 + \sin x)\, dx,$$

giving your answer correct to 2 decimal places. [3]

Answer: 0.72 J15/32/Q1

4 (i) Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{\frac{1}{6}\pi}^{\frac{2}{3}\pi} \operatorname{cosec} x\, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Using a sketch of the graph of $y = \operatorname{cosec} x$, explain whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part (i). [2]

Answers: (i) 1.95; (ii) overestimate N14/32/Q2

HOMEWORK TRAPEZIUM RULE– VARIANTS 31 & 33

1 Use the trapezium rule with three intervals to find an approximation to

$$\int_0^3 |3^x - 10| \, dx.$$ [4]

Answer: 21 31/J15/2

2 (i) Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{\frac{1}{6}\pi}^{\frac{2}{3}\pi} \operatorname{cosec} x \, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Using a sketch of the graph of $y = \operatorname{cosec} x$, explain whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part (i). [2]

Answers: (i) 1.95; (ii) overestimate 31/N14/2

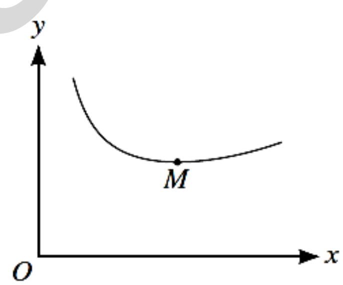
3 It is given that $I = \int_0^{0.3} (1 + 3x^2)^{-2} \, dx$.

(i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places. [3]

(ii) For small values of x, $(1 + 3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constants a and b. Hence, by evaluating $\int_0^{0.3} (1 + ax^2 + bx^4) \, dx$, find a second approximation to I, giving the answer correct to 3 decimal places. [5]

Answer: (ii) $\frac{1}{4}\left(3 - \sqrt{17}\right)$ 33/N14/Q6

4



The diagram shows a sketch of the curve $y = \dfrac{e^{\frac{1}{2}x}}{x}$ for $x > 0$, and its minimum point M.

(i) Find the x-coordinate of M. [4]

(ii) Use the trapezium rule with two intervals to estimate the value of

$$\int_1^3 \frac{e^{\frac{1}{2}x}}{x}\, dx,$$

giving your answer correct to 2 decimal places. [3]

(iii) The estimate found in part **(ii)** is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with four intervals would be greater than E or less than E. [1]

Answers: **(i)** $x = 2$ **(ii)** 2.93 **J17/33/Q7**

CHAPTER 17: DIFFERENTIAL EQUATION

Example 1

Solve $\frac{dy}{dx} = 2xy$.

Example 2

$(ii)\ \frac{dy}{dx} = y \quad (iii)\ \frac{dy}{dx} = \frac{y}{x} \quad (iv)\ \frac{dy}{dx} = -\frac{y^2}{\cos^2 x}$

Example 3

Solve $\frac{dy}{dx} = \cos^2 y$.

Example 4

Solve $\frac{dy}{dx} = y^2 \ln x$.

Example 5

The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{1+x}{2y}$.

It is given that $y = 3$ when $x = 0$.

Solve the differential equation and obtain an expression for y in terms of x.

Example 6

It is given that, for $0 < x < 300$, $\frac{dx}{dt} = \frac{x(300-x)}{600}$ and $x = 50$ when $t = 0$. Find the particular solution of the differential equation, giving your answer as an expression for x in terms of t.

Example 7

Find the curve which satisfies $y^2 \frac{dy}{dx} = x^2 e^{x^3}$ and passes through the origin.

Example 8

Find the curve which satisfies $\frac{dy}{dx} = xy^2 e^{2x}$ and passes through $(0, 1)$.

Example 9

The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. **[8]**

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Example 10

The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x + 2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. **[9]**

Cambridge International A Level Mathematics 9709 Paper 31 Q8 November 2015

Example 11

November 2014/33 Question 8

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \tfrac{1}{5}xy^{\frac{1}{2}}\sin\left(\tfrac{1}{3}x\right).$$

(i) Find the general solution, giving y in terms of x. [6]

(ii) Given that $y = 100$ when $x = 0$, find the value of y when $x = 25$. [3]

Example 12

June 2014/31 Question 4

The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2 + e^{3x}}.$$

Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

Example 13

June 2013/33 Question 8

The variables x and t satisfy the differential equation

$$t\frac{dx}{dt} = \frac{k - x^3}{2x^2},$$

for $t > 0$, where k is a constant. When $t = 1$, $x = 1$ and when $t = 4$, $x = 2$.

(i) Solve the differential equation, finding the value of k and obtaining an expression for x in terms of t. [9]

(ii) State what happens to the value of x as t becomes large. [1]

Example 14

June 2013/32 Question 8

(i) Express $\frac{1}{x^2(2x+1)}$ in the form $\frac{A}{x^2} + \frac{B}{x} + \frac{C}{2x+1}$. [4]

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x+1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

Example 15

June 2014/33 Question 5

The variables x and θ satisfy the differential equation

$$2\cos^2\theta\frac{dx}{d\theta} = \sqrt{(2x+1)},$$

and $x = 0$ when $\theta = \frac{1}{4}\pi$. Solve the differential equation and obtain an expression for x in terms of θ. [7]

Example 15

Example 17.4 *Write the differential equation for the following statements. Use k as the constant of proportionality.*

(i) *The radius, r is increasing at a constant rate.*

(ii) *The length of a metal, x increases at a rate proportional to its length.*

(iii) *When heated, the length, L of a rod increases at a rate proportional to the square of its length.*

(iv) *The number N of bacteria increases at a rate which is inversely proportional to its value.*

(v) *The temperature T of a hot object decreases at a rate proportional to its value at any time.*

(vi) *The rate of cooling of an object is proportional to the difference between the temperature of a body, θ and temperature of its surrounding, θ_s.*

(vii) *At any time t days, the birth rate of fish is equal to one hundredth of the number N of fish present. Fish are taken from the lake at the rate of R per day.*

Example 16

A tank is draining in such a way that when the height of water in the tank is h cm, it is decreasing at the rate of $0.5\sqrt{h}$ cm s^{-1}. Initially the water in the tank is at a height of 25 cm.

a) Write down a differential equation which describes this situation.

b) Solve the differential equation to find h as a function of time.

c) What is the height of the water after 10 seconds?

d) How long does it take for the water to reach a height of 5 cm?

e) Sketch a graph of the height against time for $0 \le t \le 20$.

Example 17

The spread of a disease occurs at a rate proportional to the product of the number of people infected and the number not infected. Initially 50 out of a population of 1050 are infected and the disease is spreading at a rate of 10 new cases per day.

a) If n is the number infected after t days, show that $\frac{dn}{dt} = \frac{n(1050 - n)}{5000}$.

b) Solve this differential equation to find the number of people infected after t days.

c) How long will it take for 250 people to be infected?

d) Explain why everyone in the population will eventually be infected.

Example 18

A stone falls through the air from rest and, t seconds after it was dropped, its speed v satisfies the equation $\frac{dv}{dt} = 10 - 0.2v$.

This is modelling motion under gravity at $10\,\mathrm{m\,s^{-2}}$ with air resistance proportional to speed.

a) Show that $v = 50(1 - e^{-0.2t})$.

b) Calculate the time at which the stone reaches a speed of $20\,\mathrm{m\,s^{-1}}$.

c) Sketch the graph of v against t and hence show that the velocity of the stone will never be more than $50\,\mathrm{m\,s^{-1}}$.

d) Explain what the differential equation tells us would happen if the stone was thrown downwards with a speed of $60\,\mathrm{m\,s^{-1}}$ instead of being dropped.

Example 19

Example 17.6 *The rate of increase of a variable x is proportional to $(100 - x)^2$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 10$*

(i) *Show that x satisfies the differential equation $1000\frac{dx}{dt} = (100 - x)^2$.*

(ii) *Show that the solution of this differential equation can be written in the form:*
$$x = 100 - \frac{1000}{t + 10}.$$

(iii) *Find the value of x when $t = 90$.*

(iv) *Find the value of t when $x = 50$.*

(v) *Sketch the graph of x against t and explain what happens to x when t becomes very large.*

Example 20

The size of a colony of pests, n, which fluctuates during the year, is modelled by an entomologist with the differential equation $\frac{dn}{dt} = 0.2n(0.2 - \cos t)$, where t is the number of weeks from the start of the observations. There are 400 pests in the colony initially.

Did you know?

An entomologist is a scientist who studies insects.

a) Solve the differential equation to find n in terms of t.

b) Find how many pests there are after 3 weeks.

c) Show that the number of pests reaches a minimum after approximately 9.6 days, and find the number of pests at that time.

d) Find how many pests there are after 2π weeks.

e) Explain why the model predicts that the number of pests will grow infinitely large over time.

Example 21

A model of the deflation of a sphere of radius r cm assumes that at time t seconds after the start, the rate of decrease of the surface area is proportional to the volume at that time.

When $t = 0$, $r = 20$ cm and $\frac{dr}{dt} = -3$.

a) Show that r satisfies $\frac{dr}{dt} = -0.0075r^2$.

b) Solve this differential equation, obtaining an expression for r in terms of t.

c) How long will it take for the sphere to deflate to a radius of 10 cm?

d) How much longer will it be before the radius is 5 cm?

Example 22

In a chemical reaction a substance X reacts with another substance Y. The masses of substances X and Y after time t seconds are x and y grams respectively.

It is given that $\frac{dy}{dt} = -0.5xy$ and $x = 20e^{-2t}$. Also, we know that when $t = 0$, $y = 50$.

a) Form a differential equation in y and t.

b) Solve this equation to obtain an expression for y in terms of t.

c) Find the mass of Y that remains as t gets very large.

Example 23

Example 17.7 *A reservoir has a horizontal square base of length* 10 *m. At the time* $t = 0$*, it is empty and water begins to flow into it at a rate proportional to* $\sqrt{h}$*, where* h *is the depth of water at time* t *s.*
When $h = 1$, $\dfrac{dh}{dt} = 0.1$.

(i) *Show that* h *satisfies the differential equation* $\dfrac{dh}{dt} = 0.1\sqrt{h}$.

(ii) *By solving the differential equation, express* t *in terms of* h.

(iii) *Find the time at which the depth of water reaches* 0.25 *m.*

Example 24

June 2014/32 Question 9

The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and $(1 - 0.01N)$. When $t = 0$, $N = 20$ and $\dfrac{dN}{dt} = 0.32$.

(i) Treating N and t as continuous variables, show that they satisfy the differential equation

$$\frac{dN}{dt} = 0.02N(1 - 0.01N). \qquad [1]$$

(ii) Solve the differential equation, obtaining an expression for t in terms of N. [8]

(iii) Find the time at which the population will be double its value at $t = 0$. [1]

Example 25

June 2013/31 Question 10

Liquid is flowing into a small tank which has a leak. Initially the tank is empty and, t minutes later, the volume of liquid in the tank is $V\,\text{cm}^3$. The liquid is flowing into the tank at a constant rate of $80\,\text{cm}^3$ per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to $kV\,\text{cm}^3$ per minute where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that

$$V = \frac{1}{k}(80 - 80e^{-kt}).$$ [7]

(ii) It is observed that $V = 500$ when $t = 15$, so that k satisfies the equation

$$k = \frac{4 - 4e^{-15k}}{25}.$$

Use an iterative formula, based on this equation, to find the value of k correct to 2 significant figures. Use an initial value of $k = 0.1$ and show the result of each iteration to 4 significant figures. [3]

(iii) Determine how much liquid there is in the tank 20 minutes after the liquid started flowing, and state what happens to the volume of liquid in the tank after a long time. [2]

Example 26

November 2014/31 Question 7

In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

HOMEWORK: DIFFERENTIAL EQUATION VARIANT 32

1 In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 1.9$.

(i) Show that x satisfies the differential equation

$$\frac{dx}{dt} = 0.02(100 - x).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

(iii) State what happens to the value of x as t becomes very large. [1]

Answers: **(ii)** $x = 100 - 95\exp(-0.02t)$; **(iii)** x tends to 100. J03/Q7

2 Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 1}{y^2},$$

obtaining an expression for y in terms of x. [6]

Answer: $y = (2e^{3x} - 1)^{\frac{1}{3}}$. J04/Q6

3 (i) Using partial fractions, find

$$\int \frac{1}{y(4 - y)}\, dy.$$ [4]

(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4 - y),$$

obtaining an expression for y in terms of x. [4]

(iii) State what happens to the value of y if x becomes very large and positive. [1]

Answers: **(i)** $\frac{1}{4}\ln y - \frac{1}{4}\ln(4 - y)$; **(ii)** $y = \frac{4}{3e^{-4x} + 1}$; **(iii)** The value of y tends to 4. J05/Q8

4 In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

Answer: (ii) $x = 250(3\,e^{0.1t} + 1)$ J06/Q5

5 A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9 - h)^{\frac{1}{3}}$. It is given that, when $t = 0$, $h = 1$ and $\frac{dh}{dt} = 0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt} = 0.1(9 - h)^{\frac{1}{3}}.$$ [2]

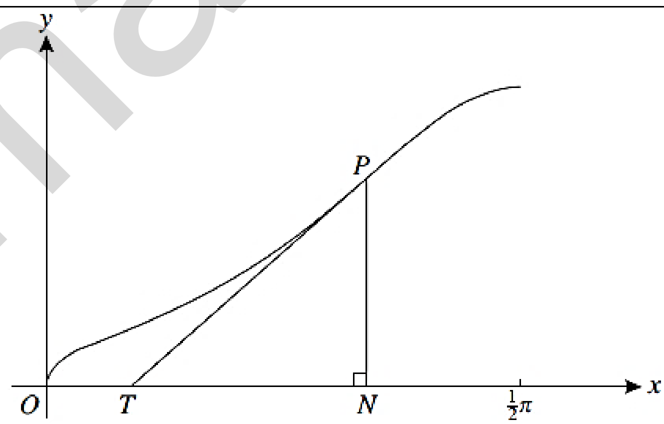
(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

Answers: (ii) $h = 9 - \left(4 - \frac{1}{15}t\right)^{\frac{3}{2}}$; (iii) 9 m, 60 years; (iv) 19.1 s. J07/Q10

6



In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.

(i) Using the fact that the gradient of the curve at P is $\frac{PN}{TN}$, show that

$$\frac{dy}{dx} = \tfrac{1}{2}y^2 \cot x.$$ [3]

(ii) Given that $y = 2$ when $x = \tfrac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x. [6]

Answer: (ii) $y = \dfrac{2}{1 - \ln(2\sin x)}$. J08/Q8

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7 **(i)** Express $\dfrac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \tfrac{1}{100}x^2(10-x),$$

obtaining an expression for t in terms of x. [6]

Answers: (i) $\dfrac{1}{x} + \dfrac{10}{x^2} + \dfrac{1}{10-x}$; (ii) $t = \ln\left(\dfrac{9x}{10-x}\right) - \dfrac{10}{x} + 10$ J09/Q8

---

8 The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

Answers: (i) $x = \tan^{-1}(\tfrac{1}{2} - \tfrac{1}{2}e^{-2t})$; (ii) x tends to $\tan^{-1}(\tfrac{1}{2})$. J10/32/Q7

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9 In an experiment to study the spread of a soil disease, an area of $10\,m^2$ of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, $5\,m^2$ was infected and the rate of growth of the infected area was $0.1\,m^2$ per day. At time t days after the start of the experiment, an area $a\,m^2$ is infected and an area $(10-a)\,m^2$ is uninfected.

(i) Show that $\dfrac{da}{dt} = 0.004a(10-a)$. [2]

(ii) By first expressing $\dfrac{1}{a(10-a)}$ in partial fractions, solve this differential equation, obtaining an expression for t in terms of a. [6]

(iii) Find the time taken for 90% of the soil area to become infected, according to this model. [2]

Answers: (ii) $t = 25\ln\left(\frac{a}{10-a}\right)$; (iii) 54.9 days. **N02/Q9**

10 Compressed air is escaping from a container. The pressure of the air in the container at time t is P, and the constant atmospheric pressure of the air outside the container is A. The rate of decrease of P is proportional to the square root of the pressure difference $(P - A)$. Thus the differential equation connecting P and t is

$$\frac{dP}{dt} = -k\sqrt{(P-A)},$$

where k is a positive constant.

(i) Find, in any form, the general solution of this differential equation. [3]

(ii) Given that $P = 5A$ when $t = 0$, and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$. [4]

(iii) Find the value of t when $P = A$. [2]

(iv) Obtain an expression for P in terms of A and t. [2]

Answers: (i) $2\sqrt{(P-A)} = -kt + c$; (iii) 4; (iv) $P = \frac{1}{4}A(4 + (4-t)^2)$. **N03/Q9**

11 A rectangular reservoir has a horizontal base of area $1000\,\text{m}^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30\,\text{m}^3\,\text{s}^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\frac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$ [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part (i) becomes

$$(x-3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]

Answers: (ii) $t = 200\,(x - 3 - 3\ln x + 3\ln 3)$; (iii) 259 s. **N04/Q10**

12 In a certain chemical reaction the amount, x grams, of a substance present is decreasing. The rate of decrease of x is proportional to the product of x and the time, t seconds, since the start of the reaction. Thus x and t satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where k is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

(i) Solve this differential equation, obtaining a relation between x, k and t. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

Answers: **(i)** $\ln x = -\frac{1}{2}kt^2 + \ln 100$; **(ii)** 51.3 s. N05/Q8

13 Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for y^2 in terms of x. [6]

Answer: $y^2 = 5e^{2x} - 1$. N06/Q4

14 The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt} = kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

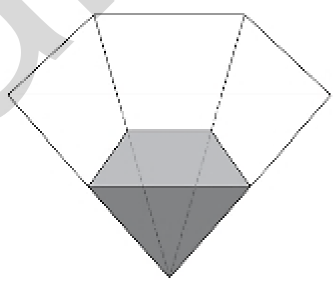
(i) Solve the differential equation, obtaining a relation between N, k and t. [5]

(ii) Given also that $N = 166$ when $t = 30$, find the value of k. [2]

(iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model. [3]

Answers: **(i)** $\ln N = 50k\sin(0.02t) + \ln 125$; **(ii)** 0.0100; **(iii)** $N = 125\exp(0.502\sin(0.02t))$, 75.6. N07/Q7

15



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\dfrac{20h^2}{100-h^2} \equiv -20 + \dfrac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part **(i)**, obtaining an expression for t in terms of h. [5]

Answer: (iii) $t = 100\ln\left(\dfrac{10+h}{10-h}\right) - 20h$. **N08/Q8**

16 The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln\frac{3}{2}$. [1]

(iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

Answers: (i) $\ln(\theta - A) = -kt + \ln 3A$; (iii) $\theta = \dfrac{7}{3}A$. **N09/32/Q9**

17 A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

Answers: **(ii)** $-\ln(20 - x) = 0.05t - \ln 20$; **(iii)** 7.9; **(iv)** x approaches 20. **N10/32/Q10**

18 A certain curve is such that its gradient at a point (x, y) is proportional to xy. At the point $(1, 2)$ the gradient is 4.

(i) By setting up and solving a differential equation, show that the equation of the curve is $y = 2e^{x^2-1}$. [7]

(ii) State the gradient of the curve at the point $(-1, 2)$ and sketch the curve. [2]

Answer: **(ii)** -4. **J11/32/Q6**

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19 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{2x+y},$$

and $y = 0$ when $x = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

Answer: $y = \ln\left(\dfrac{2}{3 - e^{2x}}\right)$ **J12/32/Q5**

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20 (i) Express $\dfrac{1}{x^2(2x+1)}$ in the form $\dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{2x+1}$. [4]

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x+1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

Answers: $\dfrac{1}{x^2} - \dfrac{2}{x} + \dfrac{4}{2x+1}$; **(ii)** $\ln y = 1 - \dfrac{1}{x} + 2\ln\left(\dfrac{2x+1}{3x}\right)$, $\dfrac{25}{36}e^{\frac{1}{2}}$ **J13/32/Q8**

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21 The population of a country at time t years is N millions. At any time, N is assumed to increase at a rate proportional to the product of N and $(1 - 0.01N)$. When $t = 0$, $N = 20$ and $\dfrac{dN}{dt} = 0.32$.

(i) Treating N and t as continuous variables, show that they satisfy the differential equation

$$\frac{dN}{dt} = 0.02N(1 - 0.01N). \quad [1]$$

(ii) Solve the differential equation, obtaining an expression for t in terms of N. [8]

(iii) Find the time at which the population will be double its value at $t = 0$. [1]

Answer: **(ii)** $t = 50\ln(4N/(100 - N))$; **(iii)** $t = 49.0$ **J14/32/Q9**

22 The number of organisms in a population at time t is denoted by x. Treating x as a continuous variable, the differential equation satisfied by x and t is

$$\frac{dx}{dt} = \frac{xe^{-t}}{k + e^{-t}},$$

where k is a positive constant.

(i) Given that $x = 10$ when $t = 0$, solve the differential equation, obtaining a relation between x, k and t. [6]

(ii) Given also that $x = 20$ when $t = 1$, show that $k = 1 - \frac{2}{e}$. [2]

(iii) Show that the number of organisms never reaches 48, however large t becomes. [2]

Answer. **(i)** $\ln x - \ln 10 = -\ln(k + e^{-t}) + \ln(k + 1)$ **J15/32/Q9**

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23 The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x + 1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

Answer: $x = \sqrt{2\sin 2\theta} - 1$ **N11/32/Q4**

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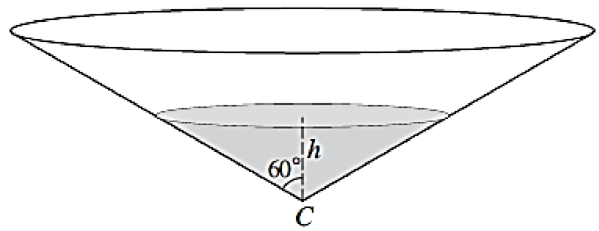
24 The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [8]

Answer. $y = \dfrac{x^2 - 4}{x^2 + 4}$. **N12/32/Q6**

25



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

(i) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part (i) and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

N13/32/Q10

Answer: **(ii)** $t = 60(1-\left(\frac{h}{H}\right)^{\frac{5}{2}})$ **(iii)** 49.4

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26 In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

Answers: **(i)** $R = 44.7xe^{-0.57x}$, **(ii)** 28.8

N14/32/Q7

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27 The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x+2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. [9]

Answer: 0.962 N15/32/Q8

HOMEWORK: DIFFERENTIAL EQUATION– VARIANTS 31 & 33

1 The number of micro-organisms in a population at time t is denoted by M. At any time the variation in M is assumed to satisfy the differential equation

$$\frac{dM}{dt} = k(\sqrt{M})\cos(0.02t),$$

where k is a constant and M is taken to be a continuous variable. It is given that when $t = 0$, $M = 100$.

(i) Solve the differential equation, obtaining a relation between M, k and t. [5]

(ii) Given also that $M = 196$ when $t = 50$, find the value of k. [2]

(iii) Obtain an expression for M in terms of t and find the least possible number of micro-organisms. [2]

33/J15/7

Answers: (i) $2\sqrt{M} = 50k\sin(0.02t) + 20$ (ii) 0.19 (iii) 27.6 or 28

2 Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = 4x(3y^2 + 10y + 3),$$

obtaining an expression for y in terms of x. [9]

31/J15/7

Answer: $y = \frac{3e^{16x^2} - 1}{3 - e^{16x^2}}$

3 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \tfrac{1}{5}xy^{\frac{1}{2}}\sin\left(\tfrac{1}{3}x\right).$$

(i) Find the general solution, giving y in terms of x. [6]

(ii) Given that $y = 100$ when $x = 0$, find the value of y when $x = 25$. [3]

33/N14/8

Answers: (i) $y = \left(-\frac{3}{10}x\cos\frac{1}{3}x + \frac{9}{10}\sin\frac{1}{3}x + c\right)^2$ (ii) 203

4 In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

Answers: (i) $R = 44.7xe^{-0.57x}$, (ii) 28.8 31/N14/7

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5 The variables x and θ satisfy the differential equation

$$2\cos^2\theta \frac{dx}{d\theta} = \sqrt{(2x+1)},$$

and $x = 0$ when $\theta = \frac{1}{4}\pi$. Solve the differential equation and obtain an expression for x in terms of θ. [7]

Answer: $x = \frac{1}{8}(\tan\theta + 1)^2 - \frac{1}{2}$ 33/J14/5

---

6 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2 + e^{3x}}.$$

Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

31/J14/4

Answer: $y = 4(2 + e^{3x})^2$

---

7 The variables x and t satisfy the differential equation

$$t\frac{dx}{dt} = \frac{k - x^3}{2x^2},$$

for $t > 0$, where k is a constant. When $t = 1$, $x = 1$ and when $t = 4$, $x = 2$.

(i) Solve the differential equation, finding the value of k and obtaining an expression for x in terms of t. [9]

(ii) State what happens to the value of x as t becomes large. [1]

33/J13/9

Answer: 0.685 *Answer*: $\frac{8}{15}$

8 Liquid is flowing into a small tank which has a leak. Initially the tank is empty and, t minutes later, the volume of liquid in the tank is $V\,\text{cm}^3$. The liquid is flowing into the tank at a constant rate of $80\,\text{cm}^3$ per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to $kV\,\text{cm}^3$ per minute where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that

$$V = \frac{1}{k}(80 - 80e^{-kt}).$$ [7]

Answer: $\frac{dV}{dt} = 80 - kV$ 31/J13/10

9 The variables x and y are related by the differential equation

$$(x^2 + 4)\frac{dy}{dx} = 6xy.$$

It is given that $y = 32$ when $x = 0$. Find an expression for y in terms of x. [6]

Answer: $y = \frac{1}{2}\left(x^2 + 4\right)^3$. 33/N12/4

10 The variables x and y are related by the differential equation

$$x\frac{dy}{dx} = 1 - y^2.$$

When $x = 2$, $y = 0$. Solve the differential equation, obtaining an expression for y in terms of x. [8]

31/N12/6

Answer: $y = \frac{x^2 - 4}{x^2 + 4}$.

11 In a certain chemical process a substance A reacts with another substance B. The masses in grams of A and B present at time t seconds after the start of the process are x and y respectively. It is given that $\frac{dy}{dt} = -0.6xy$ and $x = 5e^{-3t}$. When $t = 0$, $y = 70$.

(i) Form a differential equation in y and t. Solve this differential equation and obtain an expression for y in terms of t. [6]

(ii) The percentage of the initial mass of B remaining at time t is denoted by p. Find the exact value approached by p as t becomes large. [2]

33/J12/5

Answers: (i) $y = 70\exp(e^{-3t} - 1)$ (ii) $\frac{100}{e}$

12 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6xe^{3x}}{y^2}.$$

It is given that $y = 2$ when $x = 0$. Solve the differential equation and hence find the value of y when $x = 0.5$, giving your answer correct to 2 decimal places. [8]

Answers: (i) $y^3 = 6xe^{3x} - 2e^{3x} + 10$ (ii) 2.44 31/J12/7

13 During an experiment, the number of organisms present at time t days is denoted by N, where N is treated as a continuous variable. It is given that

$$\frac{dN}{dt} = 1.2e^{-0.02t}N^{0.5}.$$

When $t = 0$, the number of organisms present is 100.

(i) Find an expression for N in terms of t. [6]

(ii) State what happens to the number of organisms present after a long time. [1]

Answer: (i) $N = \left(40 - 30e^{-0.02t}\right)^2$; (ii) 1600 **33/N11/4**

---

14 The variables x and θ are related by the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (x+1)\cos 2\theta,$$

where $0 < \theta < \frac{1}{2}\pi$. When $\theta = \frac{1}{12}\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

Answer: $x = \sqrt{2\sin 2\theta} - 1$ **31/N11/4**

---

15 In a chemical reaction, a compound X is formed from two compounds Y and Z. The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10 - x$ and $20 - x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.01(10 - x)(20 - x).$$ [1]

(ii) Solve this differential equation and obtain an expression for x in terms of t. [9]

(iii) State what happens to the value of x when t becomes large. [1]

Answers: (ii) $x = \frac{20\left(e^{0.1t} - 1\right)}{2e^{0.1t} - 1}$; (iii) x approaches 10. **33/J11/9**

---

16 The number of birds of a certain species in a forested region is recorded over several years. At time t years, the number of birds is N, where N is treated as a continuous variable. The variation in the number of birds is modelled by

$$\frac{dN}{dt} = \frac{N(1800 - N)}{3600}.$$

It is given that $N = 300$ when $t = 0$.

(i) Find an expression for N in terms of t. [9]

(ii) According to the model, how many birds will there be after a long time? [1]

Answers: (i) $N = \dfrac{1800e^{\frac{1}{2}t}}{5+e^{\frac{1}{2}t}}$; (ii) Approaches 1800. 31/J11/10

17 A biologist is investigating the spread of a weed in a particular region. At time t weeks after the start of the investigation, the area covered by the weed is $A\,m^2$. The biologist claims that the rate of increase of A is proportional to $\sqrt{(2A-5)}$.

(i) Write down a differential equation representing the biologist's claim. [1]

(ii) At the start of the investigation, the area covered by the weed was $7\,m^2$ and, 10 weeks later, the area covered was $27\,m^2$. Assuming that the biologist's claim is correct, find the area covered 20 weeks after the start of the investigation. [9]

Answers: (i) $\dfrac{dA}{dt} = k\sqrt{2A-5}$; (ii) $63\,m^2$. 33/N10/9

18 A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(20-x)$. When $t=0$, $x=0$ and $\dfrac{dx}{dt}=1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20-x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t=10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

Answers: (ii) $-\ln(20-x) = 0.05t - \ln 20$; (iii) 7.9; (iv) x approaches 20. 31/N10/10

19 Given that $x=1$ when $t=0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{x} - \frac{x}{4},$$

obtaining an expression for x^2 in terms of t. [7]

Answer: $x^2 = 4 - 3\exp(-\frac{1}{2}t)$. 33/J10/4

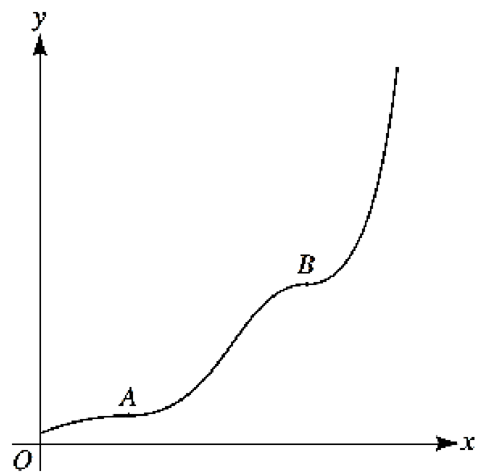
20 Given that $y=0$ when $x=1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

Answer: $y^2 = 4(x^2-1)$. 31/J10/5

21



A particular solution of the differential equation

$$3y^2\frac{dy}{dx} = 4(y^3 + 1)\cos^2 x$$

is such that $y = 2$ when $x = 0$. The diagram shows a sketch of the graph of this solution for $0 \leqslant x \leqslant 2\pi$; the graph has stationary points at A and B. Find the y-coordinates of A and B, giving each coordinate correct to 1 decimal place. [10]

Answer: 5.9 48.1 **33/N13/10**

22 Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N, where N is treated as a continuous variable.

(i) It is given that the rate of increase of N with respect to t is proportional to $(N - 150)$. Write down a differential equation relating N, t and a constant of proportionality. [1]

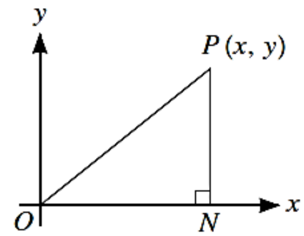
(ii) Initially, when $t = 0$, the number of plants was 650. It was noted that, at a time when there were 900 plants, the number of plants was increasing at a rate of 60 per year. Express N in terms of t. [7]

(iii) The naturalists had a target of increasing the number of plants from 650 to 2000 within 15 years. Will this target be met? [2]

N15/33/Q10

Answers: (i) $\frac{dN}{dT} = k(N - 150)$ (ii) $N = 500e^{0.08t} + 150$

23



The diagram shows a variable point P with coordinates (x, y) and the point N which is the foot of the perpendicular from P to the x-axis. P moves on a curve such that, for all $x \geqslant 0$, the gradient of the curve is equal in value to the area of the triangle OPN, where O is the origin.

(i) State a differential equation satisfied by x and y. [1]

The point with coordinates $(0, 2)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

(iii) Sketch the curve. [1]

Answer: $\frac{dy}{dx} = \frac{xy}{2}$ *Answer:* $y = 2e^{\frac{1}{4}x^2}$ **N16/33/Q5**

24 The coordinates (x, y) of a general point on a curve satisfy the differential equation

$$x\frac{dy}{dx} = (2 - x^2)y.$$

The curve passes through the point $(1, 1)$. Find the equation of the curve, obtaining an expression for y in terms of x. [7]

Answer: $y = x^2 \exp\left(\frac{1}{2} - \frac{1}{2}x^2\right)$ **N18/33/Q5**

25 The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = y(1 - 2x^2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

Answer: $y = 2xe^{(1-x^2)}$ **J16/31/Q4**

26 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = e^{-2y}\tan^2 x,$$

for $0 \leqslant x < \frac{1}{2}\pi$, and it is given that $y = 0$ when $x = 0$. Solve the differential equation and calculate the value of y when $x = \frac{1}{4}\pi$. [8]

Answer: $\frac{1}{2}e^{2y} = \tan x - x + \frac{1}{2}$; 0.179 J16/33/Q5

27 (i) Express $\dfrac{1}{x(2x+3)}$ in partial fractions. [2]

(ii) The variables x and y satisfy the differential equation

$$x(2x+3)\frac{dy}{dx} = y,$$

and it is given that $y = 1$ when $x = 1$. Solve the differential equation and calculate the value of y when $x = 9$, giving your answer correct to 3 significant figures. [7]

Answers: **(i)** $\dfrac{1}{3x} - \dfrac{2}{3(2x+3)}$ **(ii)** 1.29 J17/31/Q9

28 In a certain chemical reaction the amount, x grams, of a substance is decreasing. The differential equation relating x and t, the time in seconds since the reaction started, is

$$\frac{dx}{dt} = -kx\sqrt{t},$$

where k is a positive constant. It is given that $x = 100$ at the start of the reaction.

(i) Solve the differential equation, obtaining a relation between x, t and k. [5]

9

(ii) Given that $t = 25$ when $x = 80$, find the value of t when $x = 40$. [3]

Answer: **(i)** $\ln x = -\frac{2}{3}kt\sqrt{t} + \ln 100$ **(ii)** 64.1 J18/31/Q6

29 (i) Express $\dfrac{1}{4-y^2}$ in partial fractions. [2]

(ii) The variables x and y satisfy the differential equation

$$x\frac{dy}{dx} = 4 - y^2,$$

and $y = 1$ when $x = 1$. Solve the differential equation, obtaining an expression for y in terms of x. [6]

Answers: **(i)** $\left(\frac{1}{4}\right)\left[\frac{1}{(2+y)}+\frac{1}{(2-y)}\right]$ **(ii)** $y=\frac{2(3x^4-1)}{(3x^4+1)}$

J18/33/Q6

CHAPTER 18: NUMERICAL SOLUTIONS TO EQUATIONS

Example 1

By sketching graphs of $y = x^3$ and $y = 4 - x$, show that the equation $x^3 + x - 4 = 0$ has a root α between 1 and 2.

Example 2

Show, by calculation, that the equation $f(x) = x^5 + x - 1 = 0$ has a root α between 0 and 1.

Example 3

a By sketching a suitable pair of graphs, show that the equation $\cos x = 2x - 1$ (where x is in radians) has only one root for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

b Verify by calculation that this root lies between $x = 0.8$ and $x = 0.9$.

Example 4

The equation $x^2 + x - 3 = 0$ has a negative root, α.

a Show that this equation can be rearranged as $x = \frac{3}{x} - 1$.

b Use the iterative formula $x_{n+1} = \frac{3}{x_n} - 1$ with a starting value of $x_1 = -2.5$ to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places, where appropriate.

Example 5

The equation $e^x - 1 = 2x$ has a root $\alpha = 0$.

a Show by calculation that this equation also has a root, β, such that $1 < \beta < 2$.

b Show that this equation can be rearranged as $x = \ln(2x + 1)$.

c Use an iteration process based on the equation in **part b**, with a suitable starting value, to find β correct to 3 significant figures.

Give the result of each step of the process to 5 significant figures.

Example 6

November 2014/33 Question 9

(i) Sketch the curve $y = \ln(x+1)$ and hence, by sketching a second curve, show that the equation

$$x^3 + \ln(x+1) = 40$$

has exactly one real root. State the equation of the second curve. [3]

(ii) Verify by calculation that the root lies between 3 and 4. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{(40 - \ln(x_n + 1))},$$

with a suitable starting value, to find the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Deduce the root of the equation

$$(e^y - 1)^3 + y = 40,$$

giving the answer correct to 2 decimal places. [2]

Example 7

i By sketching a suitable pair of graphs, show that the equation $e^{2x} = 14 - x^2$ has exactly two real roots. **[3]**

ii Show by calculation that the positive root lies between 1.2 and 1.3. **[2]**

iii Show that this root also satisfies the equation $x = \frac{1}{2}\ln(14 - x^2)$. **[1]**

iv Use an iteration process based on the equation in **part iii**, with a suitable starting value, to find the root correct to 2 decimal places. Give the result of each step of the process to 4 decimal places. **[3]**

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Example 8

June 2014/33 Question 4

The equation $x = \dfrac{10}{e^{2x} - 1}$ has one positive real root, denoted by α.

(i) Show that α lies between $x = 1$ and $x = 2$. [2]

(ii) Show that if a sequence of positive values given by the iterative formula

$$x_{n+1} = \tfrac{1}{2}\ln\left(1 + \frac{10}{x_n}\right)$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Example 9

June 2013/32 Question 2

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

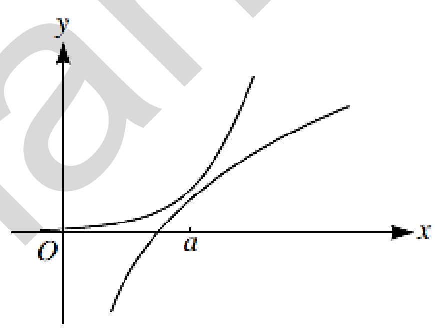
with initial value $x_1 = 3.5$, converges to α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

Example 10

June 2013/33 Question 6



The diagram shows the curves $y = e^{2x-3}$ and $y = 2\ln x$. When $x = a$ the tangents to the curves are parallel.

(i) Show that a satisfies the equation $a = \tfrac{1}{2}(3 - \ln a)$. [3]

(ii) Verify by calculation that this equation has a root between 1 and 2. [2]

(iii) Use the iterative formula $a_{n+1} = \tfrac{1}{2}(3 - \ln a_n)$ to calculate a correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

Example 11

a Show that $\left(\frac{1+\cos x}{2\sin x}\right)^2+\left(\frac{1-\cos x}{2\sin x}\right)^2=\cot^2 x+\frac{1}{2}$.

b Hence, given that α is a root of the equation $\left(\frac{1+\cos x}{2\sin x}\right)^2+\left(\frac{1-\cos x}{2\sin x}\right)^2=x$, show that α is also a root of the equation $x=\tan^{-1}\sqrt{\frac{2}{2x-1}}$ for $0<x<\frac{\pi}{2}$.

c It is given that α is the only root of the equation $x=\tan^{-1}\sqrt{\frac{2}{2x-1}}$ for $0<x<\frac{\pi}{2}$. Verify by calculation that the value of α lies between 0.9 and 1.0.

d Using an iterative formula based on the equation in **part c**, find the value of α correct to 3 significant figures.

Example 12

The equation of a curve is $y=\frac{3x^2}{x^2+4}$. At the point on the curve with positive x-coordinate p, the gradient of the curve is $\frac{1}{2}$.

i Show that $p=\sqrt{\left(\frac{48p-16}{p^2+8}\right)}$. **[5]**

ii Show by calculation that $2<p<3$. **[2]**

iii Use an iterative formula based on the equation in **part i** to find the value of p correct to 4 significant figures. Give the result of each iteration to 6 significant figures. **[3]**

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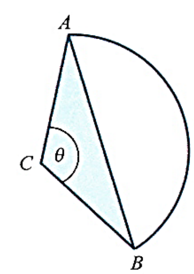
Example 13



The diagram shows the design of a company logo, ABC.
AB is an arc of a circle with centre C.

The area of the unshaded segment is to be the same as the area of the shaded triangle.

Angle ACB is θ radians.

a Show that $\theta=2\sin\theta$.

b Showing all your working, use an iterative formula based on the equation in **part a**, with an initial value of 1.85, to find θ correct to 3 significant figures.

c Hence find the length of AB, given that AC is 8 cm.

Example 14

June 2014/31 Question 8

(i) By sketching each of the graphs $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval $0 < x < \pi$. [3]

(ii) Show that the equation $\operatorname{cosec} x = x(\pi - x)$ can be written in the form $x = \dfrac{1 + x^2 \sin x}{\pi \sin x}$. [2]

(iii) The two real roots of the equation $\operatorname{cosec} x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β, where $\alpha < \beta$.

(a) Use the iterative formula

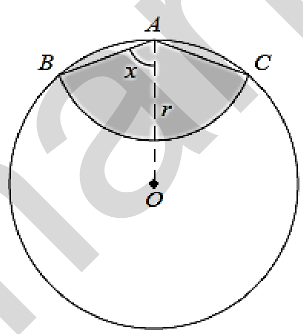
$$x_{n+1} = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$$

to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

Example 15

June 2014/32 Question 6



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is equal to x radians. The shaded region is bounded by AB, AC and the circular arc with centre A joining B and C. The perimeter of the shaded region is equal to half the circumference of the circle.

(i) Show that $x = \cos^{-1}\left(\dfrac{\pi}{4 + 4x}\right)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{\pi}{4 + 4x_n}\right)$$

to determine the value of x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Example 16

November 2014/31 Question 6

It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

HOMEWORK: NUMERICAL SOLUTIONS TO EQUATIONS VARIANT 32

1 The equation of a curve is $y = \ln x + \frac{2}{x}$, where $x > 0$.

(i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3 - \ln x_n},$$

with initial value $x_1 = 1$, converges to α. State an equation satisfied by α, and hence show that α is the x-coordinate of a point on the curve where $y = 3$. [2]

(iii) Use this iterative formula to find α correct to 2 decimal places, showing the result of each iteration. [3]

Answers: **(i)** (2, ln 2 + 1), minimum point; **(ii)** $\alpha = \frac{2}{3 - \ln \alpha}$; **(iii)** 0.56. J03/Q8

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2 (i) The equation $x^3 + x + 1 = 0$ has one real root. Show by calculation that this root lies between -1 and 0. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1 = -0.5$, to determine the root correct to 2 decimal places, showing the result of each iteration. [3]

Answer: **(iii)** –0.68. J04/Q7

---

3 (i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} x = \tfrac{1}{2}x + 1,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. [2]

(iii) Show that this root also satisfies the equation

$$x = \sin^{-1}\left(\frac{2}{x+2}\right).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2}{x_n + 2}\right),$$

with initial value $x_1 = 0.75$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iv)** 0.80 **J05/Q7**

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4 **(i)** By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0. [2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1 + e^x}\right).$$ [1]

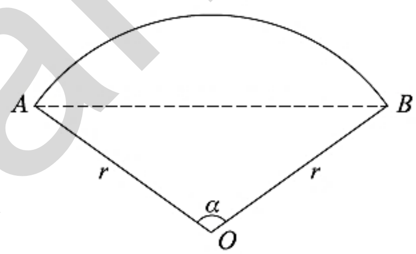
(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1 + e^{x_n}}\right),$$

with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iv)** 0.61. **J06/Q6**

5



The diagram shows a sector AOB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \pi$. The area of triangle AOB is half the area of the sector.

(i) Show that α satisfies the equation

$$x = 2\sin x.$$ [2]

(ii) Verify by calculation that α lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

(iii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \tfrac{1}{3}(x_n + 4\sin x_n)$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 1.8$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

J07/Q6

Answer: **(iv)** 1.90.

---

6 The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$. [2]

(ii) Prove that, if a sequence of values given by the iterative formula

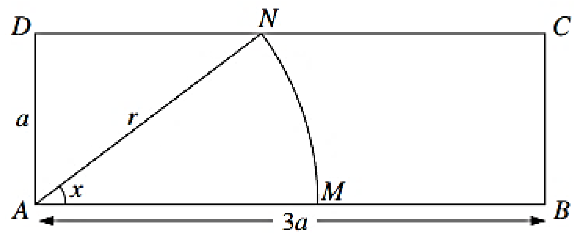
$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iii) 1.77. **J09/Q4**

7



In the diagram, $ABCD$ is a rectangle with $AB = 3a$ and $AD = a$. A circular arc, with centre A and radius r, joins points M and N on AB and CD respectively. The angle MAN is x radians. The perimeter of the sector AMN is equal to half the perimeter of the rectangle.

(i) Show that x satisfies the equation

$$\sin x = \tfrac{1}{4}(2 + x).$$ [3]

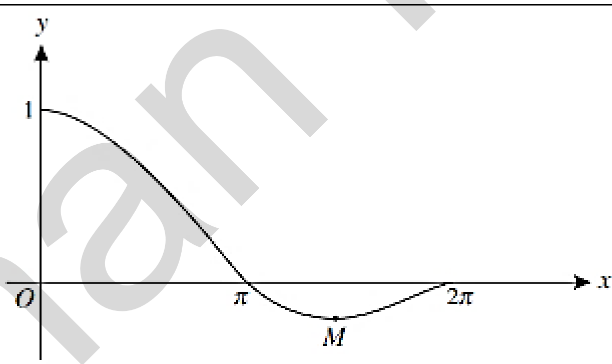
(ii) This equation has only one root in the interval $0 < x < \frac{1}{2}\pi$. Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2 + x_n}{4}\right),$$

with initial value $x_1 = 0.8$, to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer. (ii) 0.76. **J08/Q3**

8



The diagram shows the curve $y = \dfrac{\sin x}{x}$ for $0 < x \leqslant 2\pi$, and its minimum point M.

(i) Show that the x-coordinate of M satisfies the equation

$$x = \tan x.$$ [4]

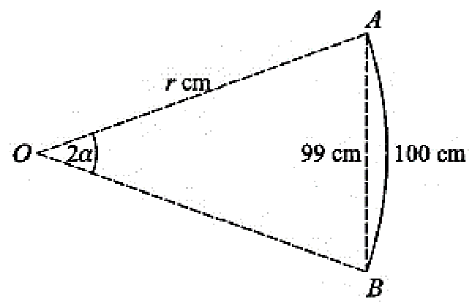
(ii) The iterative formula

$$x_{n+1} = \tan^{-1}(x_n) + \pi$$

can be used to determine the x-coordinate of M. Use this formula to determine the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer. (ii) 4.49. **J10/Q4**

9



The diagram shows a curved rod AB of length 100 cm which forms an arc of a circle. The end points A and B of the rod are 99 cm apart. The circle has radius r cm and the arc AB subtends an angle of 2α radians at O, the centre of the circle.

(i) Show that α satisfies the equation $\frac{99}{100}x = \sin x$. [3]

(ii) Given that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1} = 50\sin x_n - 48.5x_n$$

converges, then it converges to a root of the equation in part **(i)**. [2]

(iv) Use this iterative formula, with initial value $x_1 = 0.25$, to find α correct to 3 decimal places, showing the result of each iteration. [2]

Answer: **(iv)** 0.245 N02/Q7

10 **(i)** By sketching suitable graphs, show that the equation

$$\sec x = 3 - x^2$$

has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Show that, if a sequence of values given by the iterative formula

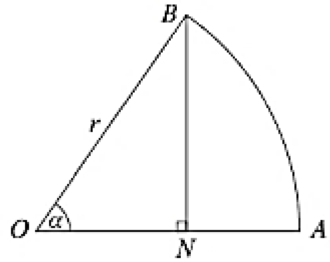
$$x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_n^2}\right)$$

converges, then it converges to a root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to determine the root in the interval $0 < x < \frac{1}{2}\pi$ correct to 2 decimal places, showing the result of each iteration. [3]

Answer: **(iii)** 1.03. N03/Q5

11



The diagram shows a sector OAB of a circle with centre O and radius r. The angle AOB is α radians, where $0 < \alpha < \frac{1}{2}\pi$. The point N on OA is such that BN is perpendicular to OA. The area of the triangle ONB is half the area of the sector OAB.

(i) Show that α satisfies the equation $\sin 2x = x$. [3]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sin(2x_n),$$

with initial value $x_1 = 1$, to find α correct to 2 decimal places, showing the result of each iteration. [3]

Answer: **(iii)** 0.95. **N04/Q5**

---

12 The equation $x^3 - x - 3 = 0$ has one real root, α.

(i) Show that α lies between 1 and 2. [2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1} = x_n^3 - 3, \qquad (A)$$

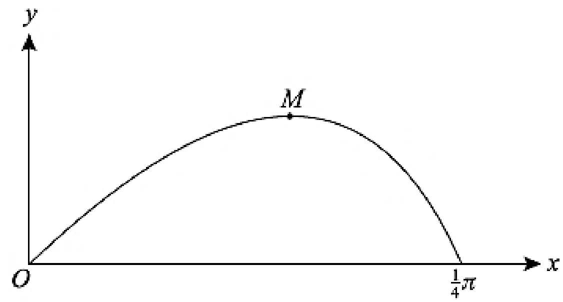
$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}. \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [5]

Answer: **(ii)** 1.67 using formula (B). **N05/Q4**

13



The diagram shows the curve $y = x\cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The point M is a maximum point.

(i) Show that the x-coordinate of M satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part (i) can be rearranged in the form $x = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)$. Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the x-axis from 0 to $\frac{1}{4}\pi$. [5]

Answers: (ii) 0.43; (iii) $\frac{1}{8}(\pi - 2)$. N06/Q10

14 **(i)** By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root. [2]

(ii) Verify by calculation that this root lies between 1.4 and 1.7. [2]

(iii) Show that this root also satisfies the equation

$$x = \tfrac{1}{3}(4 + x - 2\ln x).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \tfrac{1}{3}(4 + x_n - 2\ln x_n),$$

with initial value $x_1 = 1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iv) 1.56. N07/Q6

15 The constant a is such that $\int_0^a x e^{\frac{1}{2}x}\, dx = 6$.

(i) Show that a satisfies the equation

$$x = 2 + e^{-\frac{1}{2}x}. \qquad [5]$$

(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part (i) to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iv) 2.31 . **N08/Q9**

---

16 The equation $x^3 - 8x - 13 = 0$ has one real root.

(i) Find the two consecutive integers between which this root lies. [2]

(ii) Use the iterative formula

$$x_{n+1} = (8x_n + 13)^{\frac{1}{3}}$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answers: (i) 3, 4; (ii) 3.43. **N09/32/Q2**

---

17 **(i)** By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

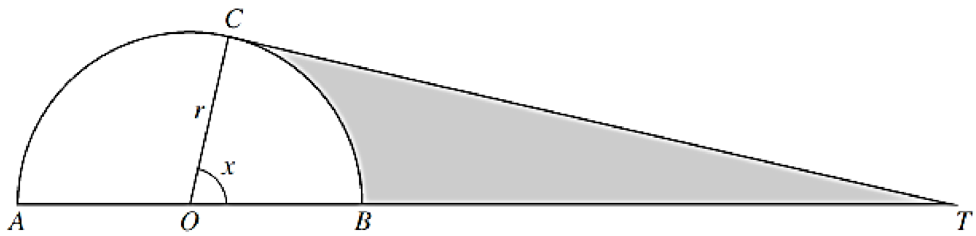
(iii) Use the iterative formula

$$x_{n+1} = \tfrac{1}{2}\sqrt{(1 + \cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iii) 0.73. **N10/32/Q4**

18



The diagram shows a semicircle ACB with centre O and radius r. The tangent at C meets AB produced at T. The angle BOC is x radians. The area of the shaded region is equal to the area of the semicircle.

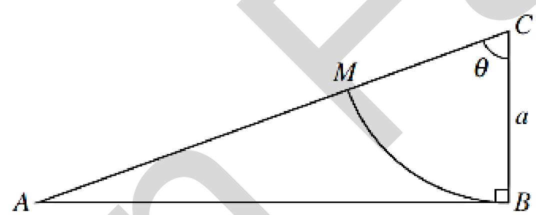
(i) Show that x satisfies the equation

$$\tan x = x + \pi. \qquad [3]$$

(ii) Use the iterative formula $x_{n+1} = \tan^{-1}(x_n + \pi)$ to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 1.35. **J11/32/Q4**

19



In the diagram, ABC is a triangle in which angle ABC is a right angle and $BC = a$. A circular arc, with centre C and radius a, joins B and the point M on AC. The angle ACB is θ radians. The area of the sector CMB is equal to one third of the area of the triangle ABC.

(i) Show that θ satisfies the equation

$$\tan\theta = 3\theta. \qquad [2]$$

(ii) This equation has one root in the interval $0 < \theta < \frac{1}{2}\pi$. Use the iterative formula

$$\theta_{n+1} = \tan^{-1}(3\theta_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 1.32 **J12/32/Q2**

20 The sequence of values given by the iterative formula

$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

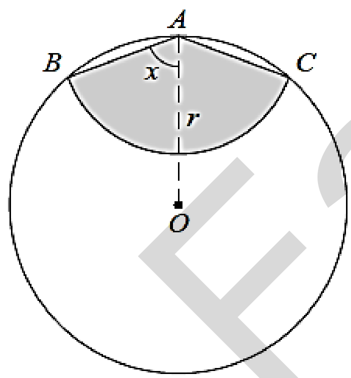
with initial value $x_1 = 3.5$, converges to α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

Answers: **(i)** 3.6840; **(ii)** $\sqrt[3]{50}$ **J13/32/Q2**

21



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is equal to x radians. The shaded region is bounded by AB, AC and the circular arc with centre A joining B and C. The perimeter of the shaded region is equal to half the circumference of the circle.

(i) Show that $x = \cos^{-1}\left(\frac{\pi}{4+4x}\right)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

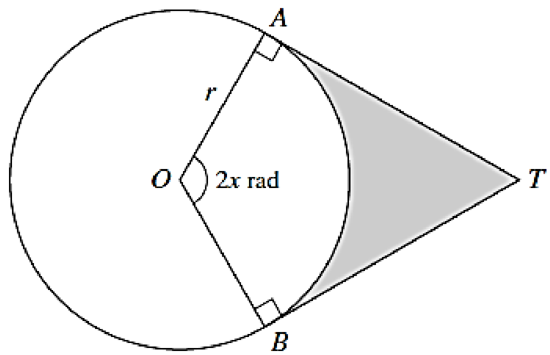
(iii) Use the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{\pi}{4+4x_n}\right)$$

to determine the value of x correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 1.21 **J14/32/Q6**

22



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and the angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The perimeter of the shaded region is equal to the circumference of the circle.

(i) Show that x satisfies the equation

$$\tan x = \pi - x.$$ [3]

(ii) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.3. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: 1.11 **J15/32/Q5**

23 (i) By sketching a suitable pair of graphs, show that the equation

$$\sec x = 3 - \tfrac{1}{2}x^2,$$

where x is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 1 and 1.4. [2]

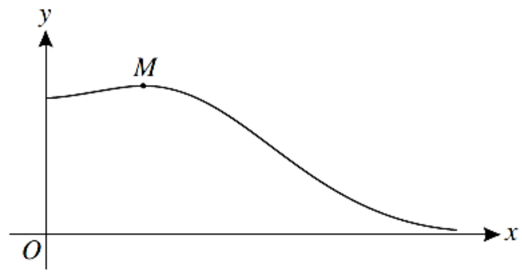
(iii) Show that this root also satisfies the equation

$$x = \cos^{-1}\left(\frac{2}{6 - x^2}\right).$$ [1]

(iv) Use an iterative formula based on the equation in part (iii) to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iv)** 1.13 **N11/32/Q5**

24



The diagram shows the curve $y = e^{-\frac{1}{2}x^2}\sqrt{(1+2x^2)}$ for $x \geqslant 0$, and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{(\ln(4+8x_n^2))},$$

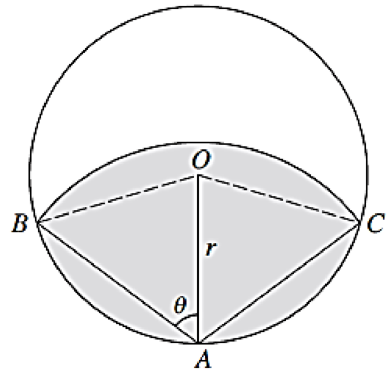
with initial value $x_1 = 2$, converges to a certain value α. State an equation satisfied by α and hence show that α is the x-coordinate of a point on the curve where $y = 0.5$. [3]

(iii) Use the iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

N12/32/Q8

Answers: **(i)** $\frac{1}{\sqrt{2}}$; **(ii)** $\alpha = \sqrt{(\ln(4+8\alpha^2))}$; **(iii)** $\alpha = 1.86$.

25



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is θ radians. The shaded region is bounded by the circumference of the circle and the arc with centre A joining B and C. The area of the shaded region is equal to half the area of the circle.

(i) Show that $\cos 2\theta = \dfrac{2\sin 2\theta - \pi}{4\theta}$. [5]

(ii) Use the iterative formula

$$\theta_{n+1} = \tfrac{1}{2}\cos^{-1}\left(\frac{2\sin 2\theta_n - \pi}{4\theta_n}\right),$$

with initial value $\theta_1 = 1$, to determine θ correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 0.95 **N13/32/Q6**

26 It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \tfrac{1}{2}\exp\left(1 + \dfrac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \tfrac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 1.94 **N14/32/Q6**

27 The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Answers: **(i)** 2, 3; **(iii)** 2.219 **N15/32/Q4**

HOMEWORK: NUMERICAL SOLUTIONS TO EQUATIONS – VARIANTS 31 & 33

1 It is given that $\int_0^a x\cos x\,dx = 0.5$, where $0 < a < \frac{1}{2}\pi$.

(i) Show that a satisfies the equation $\sin a = \dfrac{1.5 - \cos a}{a}$. [4]

(ii) Verify by calculation that a is greater than 1. [2]

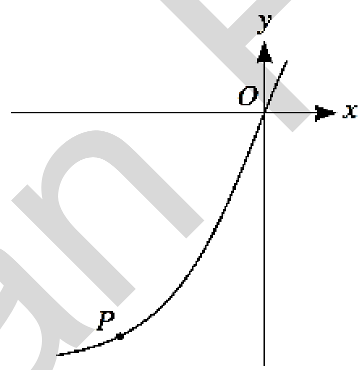
(iii) Use the iterative formula

$$a_{n+1} = \sin^{-1}\left(\frac{1.5 - \cos a_n}{a_n}\right)$$

to determine the value of a correct to 4 decimal places, giving the result of each iteration to 6 decimal places. [3]

Answers: **(iii)** 1.2461 33/J15/6

2



The diagram shows part of the curve with parametric equations

$$x = 2\ln(t+2), \qquad y = t^3 + 2t + 3.$$

(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is $\frac{1}{2}$.

(a) Show that $p = \dfrac{1}{3p^2 + 2} - 2$. [1]

(b) By first using an iterative formula based on the equation in part (a), determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and each coordinate of P correct to 2 decimal places. [4]

Answer: **(i)** $\frac{5}{2}$ **(ii)(b)** $(-5.15, -7.97)$

31/J15/10

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3 **(i)** Sketch the curve $y = \ln(x+1)$ and hence, by sketching a second curve, show that the equation

$$x^3 + \ln(x+1) = 40$$

has exactly one real root. State the equation of the second curve. [3]

(ii) Verify by calculation that the root lies between 3 and 4. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{(40 - \ln(x_n + 1))},$$

with a suitable starting value, to find the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Deduce the root of the equation

$$(e^y - 1)^3 + y = 40,$$

giving the answer correct to 2 decimal places. [2]

Answers: **(i)** $y = 40 - x^3$, **(iii)** 3.377, **(iv)** 1.48

33/N14/9

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4 It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 1.94

31/N14/6

5 The equation $x = \dfrac{10}{e^{2x} - 1}$ has one positive real root, denoted by α.

(i) Show that α lies between $x = 1$ and $x = 2$. [2]

(ii) Show that if a sequence of positive values given by the iterative formula

$$x_{n+1} = \tfrac{1}{2}\ln\left(1 + \frac{10}{x_n}\right)$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 1.14

33/J14/4

6 (i) By sketching each of the graphs $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval $0 < x < \pi$. [3]

(ii) Show that the equation $\operatorname{cosec} x = x(\pi - x)$ can be written in the form $x = \dfrac{1 + x^2 \sin x}{\pi \sin x}$. [2]

(iii) The two real roots of the equation $\operatorname{cosec} x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β, where $\alpha < \beta$.

(a) Use the iterative formula

$$x_{n+1} = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$$

to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

Answers: **(iii)** 0.66 **(iv)** 2.48.

31/J14/8

7 It is given that $\int_0^p 4xe^{-\frac{1}{2}x}\,dx = 9$, where p is a positive constant.

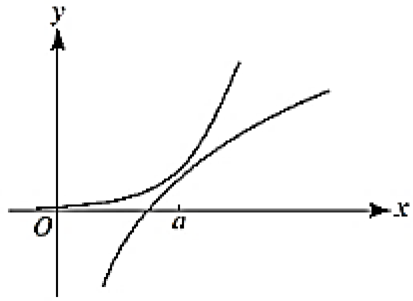
(i) Show that $p = 2\ln\left(\dfrac{8p + 16}{7}\right)$. [5]

(ii) Use an iterative process based on the equation in part **(i)** to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. [3]

Answer: **(ii)** 3.77

33/N13/5

8



The diagram shows the curves $y = e^{2x-3}$ and $y = 2\ln x$. When $x = a$ the tangents to the curves are parallel.

(i) Show that a satisfies the equation $a = \frac{1}{2}(3 - \ln a)$. [3]

(ii) Verify by calculation that this equation has a root between 1 and 2. [2]

(iii) Use the iterative formula $a_{n+1} = \frac{1}{2}(3 - \ln a_n)$ to calculate a correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

Answer: 1.35 33/J13/6

9 Liquid is flowing into a small tank which has a leak. Initially the tank is empty and, t minutes later, the volume of liquid in the tank is $V\,\text{cm}^3$. The liquid is flowing into the tank at a constant rate of $80\,\text{cm}^3$ per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to $kV\,\text{cm}^3$ per minute where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that

$$V = \frac{1}{k}(80 - 80e^{-kt}).$$ [7]

(ii) It is observed that $V = 500$ when $t = 15$, so that k satisfies the equation

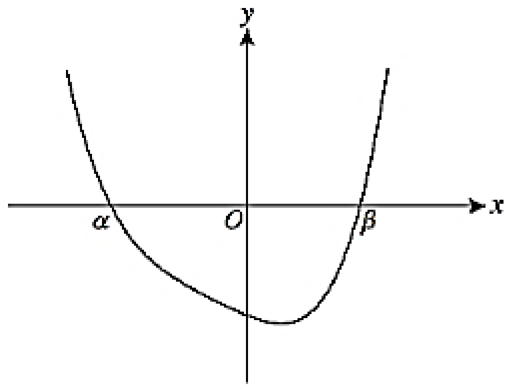
$$k = \frac{4 - 4e^{-15k}}{25}.$$

Use an iterative formula, based on this equation, to find the value of k correct to 2 significant figures. Use an initial value of $k = 0.1$ and show the result of each iteration to 4 significant figures. [3]

(iii) Determine how much liquid there is in the tank 20 minutes after the liquid started flowing, and state what happens to the volume of liquid in the tank after a long time. [2]

Answer: 0.14 *Answer*: 530 to 540, 567 to 571 31/J13/10

10



The diagram shows the curve $y = x^4 + 2x^3 + 2x^2 - 4x - 16$, which crosses the x-axis at the points $(\alpha, 0)$ and $(\beta, 0)$ where $\alpha < \beta$. It is given that α is an integer.

(i) Find the value of α. [2]

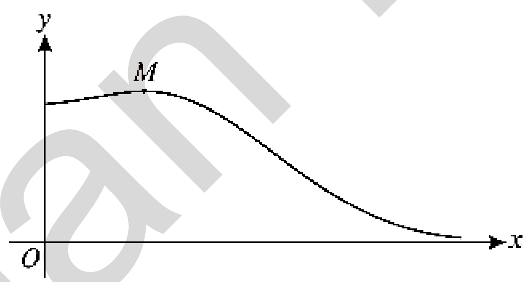
(ii) Show that β satisfies the equation $x = \sqrt[3]{(8 - 2x)}$. [3]

(iii) Use an iteration process based on the equation in part (ii) to find the value of β correct to 2 decimal places. Show the result of each iteration to 4 decimal places. [3]

33/N12/6

Answers: (i) -2; (iii) 1.67.

11



The diagram shows the curve $y = e^{-\frac{1}{2}x^2}\sqrt{(1 + 2x^2)}$ for $x \geqslant 0$, and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(\ln(4 + 8x_n^2)\right)},$$

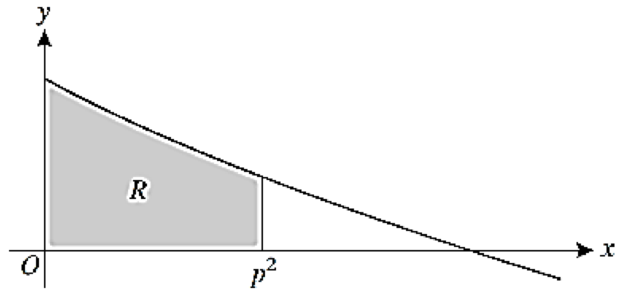
with initial value $x_1 = 2$, converges to a certain value α. State an equation satisfied by α and hence show that α is the x-coordinate of a point on the curve where $y = 0.5$. [3]

(iii) Use the iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

31/N12/8

Answers: (i) $\dfrac{1}{\sqrt{2}}$; (ii) $\alpha = \sqrt{(\ln(4 + 8\alpha^2))}$; (iii) $\alpha = 1.86$.

12



The diagram shows part of the curve $y = \cos(\sqrt{x})$ for $x \geqslant 0$, where x is in radians. The shaded region between the curve, the axes and the line $x = p^2$, where $p > 0$, is denoted by R. The area of R is equal to 1.

(i) Use the substitution $x = u^2$ to find $\int_0^{p^2} \cos(\sqrt{x})\,dx$. Hence show that $\sin p = \dfrac{3 - 2\cos p}{2p}$. [6]

(ii) Use the iterative formula $p_{n+1} = \sin^{-1}\left(\dfrac{3 - 2\cos p_n}{2p_n}\right)$, with initial value $p_1 = 1$, to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 1.25 — 33/J12/7

13 **(i)** It is given that $2\tan 2x + 5\tan^2 x = 0$. Denoting $\tan x$ by t, form an equation in t and hence show that either $t = 0$ or $t = \sqrt[3]{(t + 0.8)}$. [4]

(ii) It is given that there is exactly one real value of t satisfying the equation $t = \sqrt[3]{(t + 0.8)}$. Verify by calculation that this value lies between 1.2 and 1.3. [2]

(iii) Use the iterative formula $t_{n+1} = \sqrt[3]{(t_n + 0.8)}$ to find the value of t correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iv) Using the values of t found in previous parts of the question, solve the equation

$$2\tan 2x + 5\tan^2 x = 0$$

for $-\pi \leqslant x \leqslant \pi$. [3]

Answers: **(iii)** 1.276 **(iv)** $-\pi$, -2.24, 0, 0.906, π — 31/J12/10

14 (i) By sketching a suitable pair of graphs, show that the equation

$$\sec x = 3 - \tfrac{1}{2}x^2,$$

where x is in radians, has a root in the interval $0 < x < \tfrac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 1 and 1.4. [2]

(iii) Show that this root also satisfies the equation

$$x = \cos^{-1}\left(\frac{2}{6 - x^2}\right).$$ [1]

(iv) Use an iterative formula based on the equation in part (iii) to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iv) 1.13 31/N11/5

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15 (i) By sketching a suitable pair of graphs, show that the equation

$$\cot x = 1 + x^2,$$

where x is in radians, has only one root in the interval $0 < x < \tfrac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 0.8. [2]

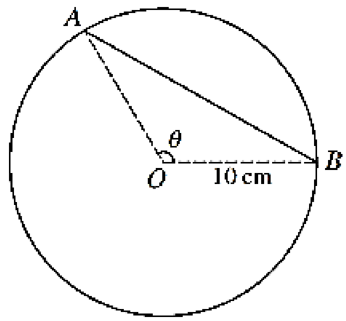
(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{1}{1 + x_n^2}\right)$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: (iii) 0.62. 33/J11/6

16



The diagram shows a circle with centre O and radius 10 cm. The chord AB divides the circle into two regions whose areas are in the ratio 1 : 4 and it is required to find the length of AB. The angle AOB is θ radians.

(i) Show that $\theta = \frac{2}{5}\pi + \sin\theta$. [3]

(ii) Showing all your working, use an iterative formula, based on the equation in part (i), with an initial value of 2.1, to find θ correct to 2 decimal places. Hence find the length of AB in centimetres correct to 1 decimal place. [5]

Answers: **(ii)** 2.11, 17.4 — 31/J11/6

17 **(i)** Given that $\int_1^a \frac{\ln x}{x^2}\,dx = \frac{2}{5}$, show that $a = \frac{5}{3}(1 + \ln a)$. [5]

(ii) Use an iteration formula based on the equation $a = \frac{5}{3}(1 + \ln a)$ to find the value of a correct to 2 decimal places. Use an initial value of 4 and give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 3.96. — 33/N10/7

18 **(i)** By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

(iii) Use the iterative formula

$$x_{n+1} = \frac{1}{2}\sqrt{(1 + \cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 0.73. — 31/N10/4

19 The curve $y = \frac{\ln x}{x+1}$ has one stationary point.

(i) Show that the x-coordinate of this point satisfies the equation

$$x = \frac{x+1}{\ln x},$$

and that this x-coordinate lies between 3 and 4. [5]

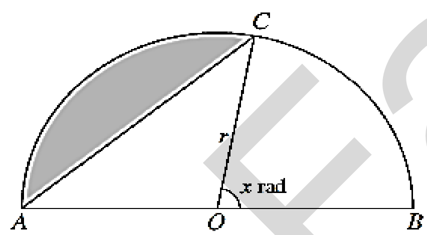
(ii) Use the iterative formula

$$x_{n+1} = \frac{x_n + 1}{\ln x_n}$$

to determine the x-coordinate correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(ii)** 3.59. **33/J10/6**

20



The diagram shows a semicircle ACB with centre O and radius r. The angle BOC is x radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that x satisfies the equation

$$x = \tfrac{3}{4}\pi - \sin x.$$ [3]

(ii) This equation has one root. Verify by calculation that the root lies between 1.3 and 1.5. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tfrac{3}{4}\pi - \sin x_n$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 1.38. **31/J10/6**

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21 A curve has parametric equations

$$x = t^2 + 3t + 1, \qquad y = t^4 + 1.$$

The point P on the curve has parameter p. It is given that the gradient of the curve at P is 4.

(i) Show that $p = \sqrt[3]{(2p+3)}$. [3]

(ii) Verify by calculation that the value of p lies between 1.8 and 2.0. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 1.89 N15/33/Q4

22 **(i)** By sketching a suitable pair of graphs, show that the equation $x^3 = 3 - x$ has exactly one real root. [2]

(ii) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Answer: **(iii)** 1.213 N18/33/Q3

23 **(i)** By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 1.43 J16/31/Q6

24 The curve with equation $y = x^2 \cos \frac{1}{2}x$ has a stationary point at $x = p$ in the interval $0 < x < \pi$.

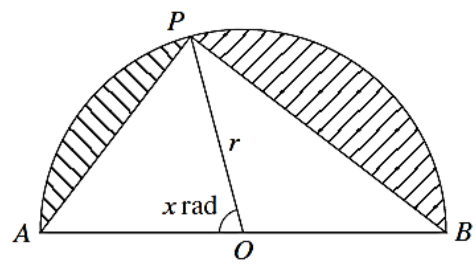
(i) Show that p satisfies the equation $\tan \frac{1}{2}p = \frac{4}{p}$. [3]

(ii) Verify by calculation that p lies between 2 and 2.5. [2]

(iii) Use the iterative formula $p_{n+1} = 2\tan^{-1}\left(\frac{4}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 2.15 J16/33/Q6

25



The diagram shows a semicircle with centre O, radius r and diameter AB. The point P on its circumference is such that the area of the minor segment on AP is equal to half the area of the minor segment on BP. The angle AOP is x radians.

(i) Show that x satisfies the equation $x = \frac{1}{3}(\pi + \sin x)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Answer: **(iii)** 1.374 J17/31/Q5

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26 The equation $\cot x = 1 - x$ has one root in the interval $0 < x < \pi$, denoted by α.

(i) Show by calculation that α is greater than 2.5. [2]

(ii) Show that, if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{1 - x_n}\right)$ converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Answer: $\alpha = 2.576$ J17/33/Q6

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27 The positive constant a is such that $\displaystyle\int_0^a x e^{-\frac{1}{2}x}\,dx = 2$.

(i) Show that a satisfies the equation $a = 2\ln(a + 2)$. [5]

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

(iii) Use an iteration based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 3.36 J18/31/Q8

28 The curve with equation $y = \dfrac{\ln x}{3+x}$ has a stationary point at $x = p$.

(i) Show that p satisfies the equation $\ln x = 1 + \dfrac{3}{x}$. [3]

(ii) By sketching suitable graphs, show that the equation in part (i) has only one root. [2]

(iii) It is given that the equation in part (i) can be written in the form $x = \dfrac{3+x}{\ln x}$. Use an iterative formula based on this rearrangement to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Answer: **(iii)** 4.97 **J18/33/Q4**

CHAPTER 19: COMPLEX NUMBERS

Example 1

Write the following numbers in their simplest form.

a $\sqrt{-9}$ b $\sqrt{-5}$ c $\sqrt{-18}$

Example 2

Solve the equation in each case.

a) $x^2 = -25$ b) $4x^2 + 484 = 0$ c) $x^2 + 20 = 0$

Example 3

Solve the equation $5z^2 + 14z + 13 = 0$.

Example 4

Solve the equation $2z^2 - z + 3 = 0$.

Example 5

Simplify:

a) i^3 b) i^4 c) i^9 d) $(-i)^6$ e) $(2i)^5$.

Example 6

Without using a calculator, find:

a $(3i^2) + (3i)^2$ b $-8i + (-4i)^3$ c $\sqrt{\dfrac{9i + 16i}{4i}}$ d i^{-6}.

Example 7

The complex numbers z_1 and z_2 are given by $z_1 = 5 + 2i$ and $z_2 = 3 + 4i$.

Find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and $\dfrac{z_1}{z_2}$.

Example 8

. Express in the form $x + iy$, where x and y are real.

a) $\dfrac{1+i}{1-i}$ b) $\dfrac{2}{3+5i}$ c) $\dfrac{2i-1}{1+2i}$ d) $\dfrac{1}{a+ib}$ e) $\dfrac{1}{4+3i} + \dfrac{12}{4-3i}$

Example 9

$(2x + y) + i(y - 5) = 0$

Find the value of x and the value of y.

Example 10

a) Find the value of the real number p such that $p + (2 - 3i)(1 + 5i)$ is an imaginary number.

b) Find the values of the real numbers x and y such that $4(x + iy) = -2y - 3ix - 5(3 + 2i)$.

Example 11

Example 18.7 *If $2 + 3i$ is a root of a quadratic equation, find*

(i) *the other root,*

(ii) *the quadratic equation.*

Example 12

Example 18.8 *If 1 and $1 + i$ are roots of a cubic equation, find*

(i) *the other root,*

(ii) *the cubic equation.*

Example 13

Example 18.9 *(i)* *Verify that $z = -1 + i$ is a root of the polynomial $x^4 + 2x^2 + 4x + 8 = 0$ and write a second complex root of the equation.*

(ii) *Find the two other roots of the equation.*

Example 14

Example 18.10 *Solve the simultaneous equations*

$u + v = i$ ...... *(1)* $u + iv = 3$ .... *(2)*

giving your answer in the form $a + bi$.

Example 15

The complex number $3 + 2i$ is a root of the quadratic equation $z^2 - (5 + 2i)z + a + bi = 0$, where a and b are real.

a) Find the values of a and b.

b) **Explain** why the two roots are not complex conjugates of each other.

Example 16

Find the square roots of the complex number $7 - 24i$.

As with any other number we expect to get two square roots of any complex number.

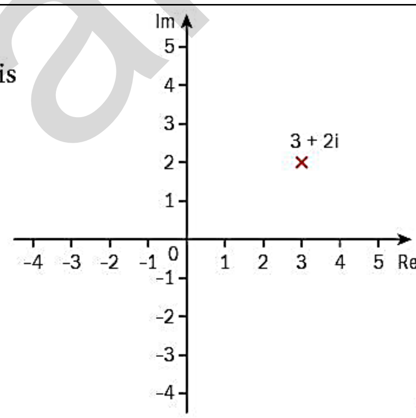
Example 17

Given that $2 + i$ is a root of the equation $z^3 - z^2 - 7z + 15 = 0$, find the other two roots.

Representing Complex Numbers Geometrically

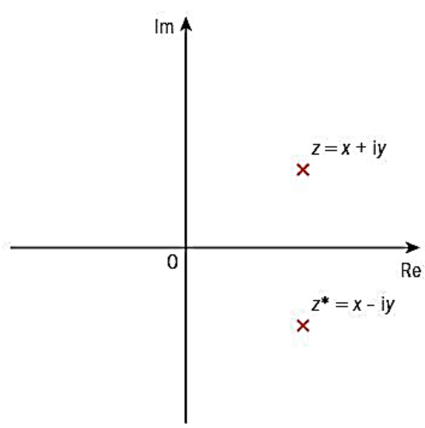
Any complex number $z = x + iy$ can be represented by using a two-dimensional set of coordinate axes where the horizontal axis is used as the 'real axis' (Re) and the vertical axis is used as the 'imaginary axis' (Im).

So, for example, the complex number $3 + 2i$ is represented by the point $(3, 2)$, as shown in the diagram.



Such a diagram is called an **Argand diagram** and is named after the Swiss mathematician Jean-Robert Argand (1768–1822).

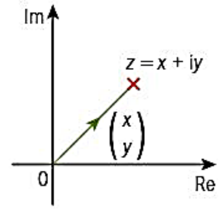
It follows that the point representing $z^* = x - iy$ is the reflection in the real axis of the point representing $z = x + iy$.



The position of the point representing z^*, the complex conjugate of z, on the Argand diagram is found by reflecting the point representing z in the real axis.

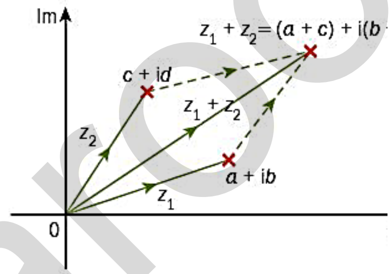
Geometrical Interpretation of operations on complex numbers

We will now consider how to represent the addition and subtraction of two complex numbers on an Argand diagram. To do this, it is helpful to consider the complex number $z = x + \mathrm{i}y$ as a point in the Argand diagram with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$.



If $z_1 = a + \mathrm{i}b$ and $z_2 = c + \mathrm{i}d$, then

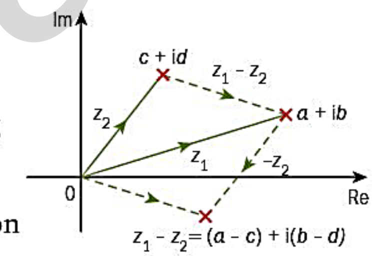
$$\begin{aligned} z_1 + z_2 &= (a + \mathrm{i}b) + (c + \mathrm{i}d) \\ &= (a + c) + \mathrm{i}(b + d) \end{aligned}$$



The point representing $z_1 + z_2$ can be constructed by adding the vectors representing z_1 and z_2; that is, the point representing $z_1 + z_2$ can be found by drawing the diagonal of the parallelogram formed by the vectors representing z_1 and z_2. This is illustrated on the diagram.

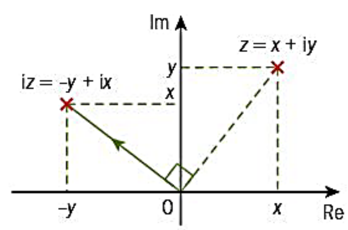
Similarly,

$$z_1 - z_2 = (a - c) + \mathrm{i}(b - d)$$



The point representing $z_1 - z_2$ can be constructed by adding the vectors representing z_1 and $-z_2$, then drawing the vector representing $z_1 - z_2$ from the origin.

We will look at how we can represent the multiplication and division of complex numbers geometrically later in the chapter. However, at this point, it is worth considering what happens when we multiply a complex number by i.



If $z = x + \mathrm{i}y$, then $\mathrm{i}z = \mathrm{i}(x + \mathrm{i}y) = \mathrm{i}x - y$, and so the point (x, y) representing z is mapped onto the point $(-y, x)$. The diagram shows that this is equivalent to a rotation of 90° anticlockwise about the origin.

> Multiplying a complex number z by the imaginary number i is geometrically equivalent to rotating the point representing z by 90° anticlockwise about the origin.

Modulus and Argument of a complex number

The **modulus of a complex number** $x + \mathrm{i}y$ is the magnitude of the position vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

Therefore, the modulus of $x + \mathrm{i}y$ is defined to be $\sqrt{x^2 + y^2}$.

Notation: The modulus of $z = x + \mathrm{i}y$ is $|z|$.

Also note that the plural of modulus is moduli.

The **argument of a complex number** $x + \mathrm{i}y$ is the direction of the position vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

More precisely, it is the angle made between the positive real axis and the position vector. The **principal argument**, θ, of a complex number is an angle such that $-\pi < \theta \leqslant \pi$. Sometimes it might be more convenient to give θ as an angle such that $0 \leqslant \theta < 2\pi$. Usually, the argument is given in radians.

Notation: The argument of $z = x + \mathrm{i}y$ is $\arg z$.

We can use trigonometry to find the argument. For example, when θ is acute, $\tan\theta = \dfrac{y}{x}$.

A diagram **must** be drawn first to check the position of the complex number.

Example 18

Find the modulus and argument of each of the following complex numbers.

a $5 + 12\mathrm{i}$ **b** $-3 + 4\mathrm{i}$ **c** $12 - 5\mathrm{i}$ **d** $-4 - 3\mathrm{i}$

Polar Form

Using trigonometry, we can now write a complex number in modulus-argument form.

$\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{x}{r}$.

Therefore, $x = r\cos\theta$ and $y = r\sin\theta$.

Example 19

$u = 6 - 3\text{i}$ and $w = -7 + 5\text{i}$

Write each of u and w in modulus-argument form.

Example 20

a) The complex numbers z_1 and z_2 are defined by $z_1 = 1 + \text{i}$ and $z_2 = -2 - 4\text{i}$. Find the modulus and argument of z_1 and z_2.

b) The complex numbers z_3 and z_4 are defined by $z_3 = \left(5, \frac{\pi}{3}\right)$ and $z_4 = \left(2, \frac{5\pi}{6}\right)$. Write z_3 and z_4 in the form $x + \text{i}y$, where x and y are real numbers.

Exponential Form

A complex number written in the form $r\text{e}^{\text{i}\theta}$, where r is the modulus and θ is the argument, is said to be in exponential form.

This is the basis for another representation, called the **exponential form**.

$$z = r(\cos\theta + \text{i}\sin\theta)$$

$$z = r\text{e}^{\text{i}\theta}$$

where r is $|z|$ and θ is $\arg z$.

The coordinates (r, θ), where r is $|z|$ and θ is $\arg z$, are called the **polar coordinates**.

When a complex number is written as $z = r\text{e}^{\text{i}\theta}$, where r is $|z|$ and θ is $\arg z$, we say it is in exponential form.

The modulus-argument and exponential forms of a complex number are **polar forms**.
A polar form uses polar coordinates; a **Cartesian form** uses Cartesian coordinates.

Example 21

$z_1 = 1 + \text{i}$ $\qquad z_2 = 5\text{e}^{\frac{\text{i}\pi}{3}}$ $\qquad z_3 = 2\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$

a Write z_1 in
- modulus-argument form
- exponential form.

b Write z_2 in
- modulus-argument form
- Cartesian form.

c Write z_3 in
- exponential form
- Cartesian form.

If $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$, then $z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$.

If $z = (r, \theta)$, then $z^n = (r^n, n\theta)$.

If $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$, then $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right)$.

Example 22

The complex numbers z and w are defined by $z = 2 + 2\text{i}$ and $w = -1 + \sqrt{3}\,\text{i}$.
Find the modulus and argument of

a) z^2 **b)** z^3 **c)** z^2w^2 **d)** $\frac{w^2}{z^3}$.

Loci of Complex Numbers

Case 1 $|z - z_1| = r$, where z_1 is a known complex number and r is real

If $|z - z_1| = r$, then z lies on a circle, centre z_1, radius r.

We can also see that if $|z - z_1| < r$, then z lies anywhere inside a circle, centre z_1, radius r;
and if $|z - z_1| > r$, then z lies anywhere outside a circle, centre z_1, radius r.

Example 23

On separate Argand diagrams, sketch these loci.

a $|z - (2 + 4\text{i})| = 3$ **b** $|z - 3\text{i}| < 4$ **c** $|z + 5| \leqslant 5$

Example 24

a) Sketch an Argand diagram and show all the points which represent the complex numbers z that satisfy the equation $|z - 1 + \text{i}| = 2$.

b) The points in an Argand diagram representing $2 + 5\text{i}$ and $-6 + \text{i}$ are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z - (a + b\text{i})| = k$.

Example 25

Sketch an Argand diagram and shade the region whose points represent the complex numbers z which satisfy both the inequality $|z - (2 + 3\text{i})| \leq 2$ and the inequality $|z - 4| > 3$.

Case 2 $|z - z_1| = |z - z_2|$ where z_1 and z_2 are known complex numbers

If $|z - z_1| = |z - z_2|$, then z lies on the perpendicular bisector of the line joining z_1 to z_2.

We can also see that if $|z - z_1| < |z - z_2|$, then z lies anywhere in the region on one side of the perpendicular bisector of the line joining z_1 to z_2 such that the distance from z to z_1 is less than the distance from z to z_2.

- $|z - z_1| < |z - z_2|$ is all points such that P is nearer to Q than to R. The perpendicular bisector marks the boundary of this region but is not included.
- $|z - z_1| \leqslant |z - z_2|$ is all points such that P is nearer to Q than to R or equidistant from Q and R. The perpendicular bisector marks the boundary of this region and is included.

Example 26

On an Argand diagram, sketch the locus $|z - 4 + \mathrm{i}| = |z + 5\mathrm{i}|$.

Find the Cartesian equation of this locus.

Example 27

a) Sketch on an Argand diagram the loci given by $|z - (1 + 2\mathrm{i})| = 5$ and $|z - 5 + \mathrm{i}| = |z + 3 - 5\mathrm{i}|$.
b) Show that these loci intersect at the point $-2 - 2\mathrm{i}$.

Case 3 $\arg(z - z_1) = \theta$ where z_1 is a known complex number and θ is an angle, measured in radians

If $\arg(z - z_1) = \theta$, then the locus of z is a half-line, starting at z_1, at an angle θ with the positive real axis.

Example 28

On a single Argand diagram, sketch the loci $|z| = 4$ and $\arg(z + 2 + 3\mathrm{i}) = \frac{\pi}{4}$.

Show that there is only one complex number, z, that satisfies both loci.

Label this point as P on your diagram.

Example 29

a) Sketch on an Argand diagram the locus of the complex number z where z satisfies the equation $\arg(z + 2 - \mathrm{i}) = \frac{2\pi}{3}$.

b) On the same diagram, shade the region whose points represent the complex number z which satisfies $\frac{\pi}{2} \le \arg(z + 2 - \mathrm{i}) \le \frac{2\pi}{3}$.

Example 30

a) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z + 4 - 4\sqrt{3}\mathrm{i}| \le 4$.

b) i) Find the least value of $|z|$ in this region.

ii) Find the greatest value of $\arg z$ in this region.

Example 31

i Without using a calculator, solve the equation

$$3w + 2\mathrm{i}w^* = 17 + 8\mathrm{i},$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + b\mathrm{i}$. **[4]**

ii In an Argand diagram, the loci

$$\arg(z - 2\mathrm{i}) = \frac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3\mathrm{i}|$$

intersect at the point P. Express the complex number represented by P in the form $re^{\mathrm{i}\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q7 June 2013

Example 32

Throughout this question the use of a calculator is not permitted.

i The complex numbers u and v satisfy the equations

$$u + 2v = 2\mathrm{i} \text{ and } \mathrm{i}u + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + \mathrm{i}y$, where x and y are real. **[5]**

ii On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \mathrm{i}| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. **[5]**

Cambridge International A Level Mathematics 9709 Paper 31 Q8 November 2013

Example 33

Throughout this question the use of a calculator is not permitted.

The complex numbers w and z satisfy the relation

$$w = \frac{z + \mathrm{i}}{\mathrm{i}z + 2}.$$

i Given that $z = 1 + \mathrm{i}$, find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. **[4]**

ii Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. **[4]**

Cambridge International A Level Mathematics 9709 Paper 31 Q5 November 2014

Example 34

The complex number $3 - \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

i On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? **[4]**

ii Showing your working and without using a calculator, express $\frac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. **[3]**

iii By considering the argument of $\frac{u^*}{u}$, prove that

$$\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right).$$ **[3]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 November 2015

Example 35

The complex number $1 + (\sqrt{2})\mathrm{i}$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by $\mathrm{p}(x)$.

i Showing your working, verify that u is a root of the equation $\mathrm{p}(x) = 0$, and write down a second complex root of the equation. **[4]**

ii Find the other two roots of the equation $\mathrm{p}(x) = 0$. **[6]**

Cambridge International A Level Mathematics 9709 Paper 31 Q9 November 2012

Example 36

June 2014/32 Question 7

(a) It is given that $-1 + (\sqrt{5})\mathrm{i}$ is a root of the equation $z^3 + 2z + a = 0$, where a is real. Showing your working, find the value of a, and write down the other complex root of this equation. [4]

(b) The complex number w has modulus 1 and argument 2θ radians. Show that $\dfrac{w-1}{w+1} = \mathrm{i}\tan\theta$. [4]

Example 37

November 2014/31 Question 5

Throughout this question the use of a calculator is not permitted.

The complex numbers w and z satisfy the relation

$$w = \frac{z+\mathrm{i}}{\mathrm{i}z+2}.$$

(i) Given that $z = 1 + \mathrm{i}$, find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

Example 38

November 2014/33 Question 5

The complex numbers w and z are defined by $w = 5 + 3\mathrm{i}$ and $z = 4 + \mathrm{i}$.

(i) Express $\dfrac{\mathrm{i}w}{z}$ in the form $x + \mathrm{i}y$, showing all your working and giving the exact values of x and y. [3]

(ii) Find wz and hence, by considering arguments, show that

$$\tan^{-1}\left(\tfrac{3}{5}\right) + \tan^{-1}\left(\tfrac{1}{4}\right) = \tfrac{1}{4}\pi. \quad [4]$$

Example 39

June 2014/33 Question 7

(a) The complex number $\dfrac{3-5\mathrm{i}}{1+4\mathrm{i}}$ is denoted by u. Showing your working, express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) (i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 - \mathrm{i}| \leqslant 1$ and $|z - \mathrm{i}| \leqslant |z - 2|$. [4]

(ii) Calculate the maximum value of $\arg z$ for points lying in the shaded region. [2]

Example 40
June 2013/31 Question 7

(a) Without using a calculator, solve the equation

$$3w + 2\mathrm{i}w^* = 17 + 8\mathrm{i},$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + b\mathrm{i}$. [4]

(b) In an Argand diagram, the loci

$$\arg(z - 2\mathrm{i}) = \tfrac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3\mathrm{i}|$$

intersect at the point P. Express the complex number represented by P in the form $re^{\mathrm{i}\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

Example 41
June 2013/32 Question 9

(a) The complex number w is such that $\operatorname{Re} w > 0$ and $w + 3w^* = \mathrm{i}w^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2\mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

Example 42
June 2013/33 Question 7

The complex number z is defined by $z = a + \mathrm{i}b$, where a and b are real. The complex conjugate of z is denoted by z^*.

(i) Show that $|z|^2 = zz^*$ and that $(z - k\mathrm{i})^* = z^* + k\mathrm{i}$, where k is real. [2]

In an Argand diagram a set of points representing complex numbers z is defined by the equation $|z - 10\mathrm{i}| = 2|z - 4\mathrm{i}|$.

(ii) Show, by squaring both sides, that

$$zz^* - 2\mathrm{i}z^* + 2\mathrm{i}z - 12 = 0.$$

Hence show that $|z - 2\mathrm{i}| = 4$. [5]

(iii) Describe the set of points geometrically. [1]

HOMEWORK: COMPLEX NUMBERS VARIANT 32

1 The complex number $2\mathrm{i}$ is denoted by u. The complex number with modulus 1 and argument $\frac{2}{3}\pi$ is denoted by w.

(i) Find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers w, uw and $\frac{u}{w}$. [4]

(ii) Sketch an Argand diagram showing the points U, A and B representing the complex numbers u, uw and $\frac{u}{w}$ respectively. [2]

(iii) Prove that triangle UAB is equilateral. [2]

J03/Q5

Answers: (i) $\cos\frac{2}{3}\pi + \mathrm{i}\sin\frac{2}{3}\pi$, $-\sqrt{3} - \mathrm{i}$, $\sqrt{3} - \mathrm{i}$.

2 The complex number u is given by $u = \frac{7+4\mathrm{i}}{3-2\mathrm{i}}$.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the complex number z such that $|z - u| = 2$. [3]

(iii) Find the greatest value of arg z for points on this locus. [3]

N03/Q7

Answers: (i) $1 + 2\mathrm{i}$; (iii) $126.9°$.

3 (i) Find the roots of the equation $z^2 - z + 1 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Obtain the modulus and argument of each root. [3]

(iii) Show that each root also satisfies the equation $z^3 = -1$. [2]

J04/Q8

Answers: (i) $\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$, $\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$; (ii) 1, $\frac{1}{3}\pi$, 1, $-\frac{1}{3}\pi$.

4 The complex numbers $1 + 3\mathrm{i}$ and $4 + 2\mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - v$ and $\frac{u}{v}$. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram, with origin O, the points A, B and C represent the numbers u, v and $u - v$ respectively.

(iii) State fully the geometrical relationship between OC and BA. [2]

(iv) Prove that angle $AOB = \frac{1}{4}\pi$ radians. [2]

Answers: **(i)** $-3 + \mathrm{i}$, $\frac{1}{2} + \frac{1}{2}\mathrm{i}$; **(ii)** $\frac{1}{4}\pi$; **(iii)** *OC* and *BA* are equal and parallel. **N04/Q6**

5 **(i)** Solve the equation $z^2 - 2\mathrm{i}z - 5 = 0$, giving your answers in the form $x + \mathrm{i}y$ where x and y are real. [3]

(ii) Find the modulus and argument of each root. [3]

(iii) Sketch an Argand diagram showing the points representing the roots. [1]

Answers: **(i)** $2 + \mathrm{i}$, $-2 + \mathrm{i}$; **(ii)** 0.464 (or 26.6°), 2.68 (or 153.4°). **J05/Q3**

6 The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that $1 + 2\mathrm{i}$ is one of the complex roots. [3]

(ii) Write down the other complex root of the equation. [1]

(iii) Sketch an Argand diagram showing the point representing the complex number $1 + 2\mathrm{i}$. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2\mathrm{i}|.$$ [4]

Answer: **(ii)** $1 - 2\mathrm{i}$. **N05/Q7**

7 The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C. [4]

(ii) Express $\frac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\frac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

Answers: **(i)** *OACB* is a rhombus (or equivalent); **(ii)** $\frac{3}{5} + \frac{4}{5}\mathrm{i}$. **J06/Q7**

8 The complex number u is given by

$$u = \frac{3+i}{2-i}.$$

(i) Express u in the form $x + iy$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

Answers: (i) $1 + i$; (ii) $\sqrt{2}$, 45°; (iv) $\sqrt{2} - 1$. N06/Q9

9 The complex number $\frac{2}{-1+i}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

Answers: (i) $\sqrt{2}$ and $-\frac{3}{4}\pi$, 2 and $\frac{1}{2}\pi$. J07/Q8

10 **(a)** The complex number z is given by $z = \frac{4-3i}{1-2i}$.

(i) Express z in the form $x + iy$, where x and y are real. [2]

(ii) Find the modulus and argument of z. [2]

(b) Find the two square roots of the complex number $5 - 12i$, giving your answers in the form $x + iy$, where x and y are real. [6]

Answers: (a)(i) $2 + i$, (ii) $\sqrt{5}$ or 2.24, 0.464 or 26.6°; (b) $-3 + 2i$, $3 - 2i$. N07/Q8

11 The variable complex number z is given by

$$z = 2\cos\theta + i(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - i| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\frac{1}{z+2-i}$ is constant for $-\pi < \theta < \pi$. [4]

12 The complex number w is given by $w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\frac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\frac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2i$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + iy$, where x and y are real and exact. [4]

Answers: **(i)** 1, $\frac{2}{3}\pi$ (or 2.09 radians); **(ii)** R, $\theta + \frac{2}{3}\pi$; R, $\theta - \frac{2}{3}\pi$;
(iv) $-(2+\sqrt{3})+(2\sqrt{3}-1)i$, $-(2-\sqrt{3})-(2\sqrt{3}+1)i$.

N08/Q10

13 **(i)** Solve the equation $z^2 + (2\sqrt{3})iz - 4 = 0$, giving your answers in the form $x + iy$, where x and y are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

Answers: **(i)** $1-\sqrt{3}i$, $-1-\sqrt{3}i$; **(iii)** 2, –60°; 2, –120°.

J09/Q7

14 The complex numbers $-2+i$ and $3+i$ are denoted by u and v respectively.

(i) Find, in the form $x+iy$, the complex numbers

(a) $u+v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u+v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

Answers: (i)(a) $1+2i$, (b) $-\frac{1}{2}+\frac{1}{2}i$; (ii) $\frac{3}{4}\pi$; (iv) $OA = BC$, OA is parallel to BC. N09/32/Q7

15 The variable complex number z is given by

$$z = 1 + \cos 2\theta + i\sin 2\theta,$$

where θ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of z is $2\cos\theta$ and the argument of z is θ. [6]

(ii) Prove that the real part of $\frac{1}{z}$ is constant. [3]

J10/Q8

16 (a) Find the two square roots of the complex number $-3+4i$, giving your answers in the form $x+iy$, where x and y are real. [5]

(b) The complex number z is given by

$$z = \frac{-1+3i}{2+i}.$$

(i) Express z in the form $x+iy$, where x and y are real. [2]

(ii) Show on a sketch of an Argand diagram, with origin O, the points A, B and C representing the complex numbers $-1+3i$, $2+i$ and z respectively. [1]

(iii) State an equation relating the lengths OA, OB and OC. [1]

Answers: (a) $1+2i$ and $-1-2i$; (b)(i) $\frac{1}{5}+\frac{7}{5}i$, (iii) $OC = \frac{OA}{OB}$. N02/Q8

17 The complex number z is given by

$$z = (\sqrt{3}) + \mathrm{i}.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x + \mathrm{i}y$, where x and y are real,

(a) $2z + z^*$,

(b) $\dfrac{\mathrm{i}z^*}{z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and $\mathrm{i}z^*$ respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

Answers: (i) 2, $\frac{1}{6}\pi$; (ii)(a) $3\sqrt{3} + \mathrm{i}$, (b) $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$. **N10/32/Q6**

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18 (a) The complex number u is defined by $u = \dfrac{5}{a + 2\mathrm{i}}$, where the constant a is real.

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Find the value of a for which $\arg(u^*) = \frac{3}{4}\pi$, where u^* denotes the complex conjugate of u. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| < 2$ and $|z| < |z - 2 - 2\mathrm{i}|$. [4]

Answers: **(a)(i)** $\dfrac{5a}{a^2+4} - \dfrac{10\,\mathrm{i}}{a^2+4}$, **(ii)** -2. **J11/32/Q7**

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19 **Throughout this question the use of a calculator is not permitted.**

The complex number u is defined by

$$u = \frac{1 + 2\mathrm{i}}{1 - 3\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Show on a sketch of an Argand diagram the points A, B and C representing the complex numbers u, $1 + 2\mathrm{i}$ and $1 - 3\mathrm{i}$ respectively. [2]

(iii) By considering the arguments of $1 + 2\mathrm{i}$ and $1 - 3\mathrm{i}$, show that

$$\tan^{-1} 2 + \tan^{-1} 3 = \tfrac{3}{4}\pi.$$ [3]

Answer: **(i)** $u = -\dfrac{1}{2} + \dfrac{1}{2}\mathrm{i}$ **J12/32/Q7**

20 **(a)** The complex number w is such that $\mathrm{Re}\,w > 0$ and $w + 3w^* = \mathrm{i}w^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2\mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

Answers: **(i)** $2\sqrt{2} - 2\,\mathrm{i}$; **(ii)** 3.70 J13/32/Q9

21 **(a)** It is given that $-1 + (\sqrt{5})\mathrm{i}$ is a root of the equation $z^3 + 2z + a = 0$, where a is real. Showing your working, find the value of a, and write down the other complex root of this equation. [4]

(b) The complex number w has modulus 1 and argument 2θ radians. Show that $\dfrac{w-1}{w+1} = \mathrm{i}\tan\theta$. [4]

Answer: **(a)** $a = -12$; $z = -1 - (\sqrt{5})\mathrm{i}$ J14/32/Q7

22 The complex number u is given by $u = -1 + (4\sqrt{3})\mathrm{i}$.

(i) Without using a calculator and showing all your working, find the two square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the real numbers a and b are exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z - u| = 1$. Determine the greatest value of $\arg z$ for points on this locus. [4]

Answers: **(i)** $\pm(\sqrt{3} + 2\mathrm{i})$; **(ii)** 1.86 J15/32/Q7

23 **(a)** Showing your working, find the two square roots of the complex number $1 - (2\sqrt{6})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3\mathrm{i}| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

Answer: **(i)** $\pm(\sqrt{3} - \mathrm{i}\sqrt{2})$ **(ii)** 131.8° (or 2.30 radians) N11/32/Q10

24 The complex number $1 + (\sqrt{2})\mathrm{i}$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by $\mathrm{p}(x)$.

(i) Showing your working, verify that u is a root of the equation $\mathrm{p}(x) = 0$, and write down a second complex root of the equation. [4]

(ii) Find the other two roots of the equation $\mathrm{p}(x) = 0$. [6]

Answers: **(i)** $1 - \sqrt{2}\,\mathrm{i}$; **(ii)** $-1 + \mathrm{i}$, $-1 - \mathrm{i}$. N12/32/Q9

25 **Throughout this question the use of a calculator is not permitted.**

(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2\text{i} \quad \text{and} \quad \text{i}u + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + \text{i}y$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \text{i}| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

Answers: **(a)** $u = -2 - 2\text{i}$, $v = 1 + 2\text{i}$ **(b)** $\frac{3}{\sqrt{2}} - 1$

N13/32/Q8

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26 **Throughout this question the use of a calculator is not permitted.**

The complex numbers w and z satisfy the relation

$$w = \frac{z + \text{i}}{\text{i}z + 2}.$$

(i) Given that $z = 1 + \text{i}$, find w, giving your answer in the form $x + \text{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \text{i}y$, where x and y are real. [4]

Answers: **(i)** $\frac{3}{2} + \frac{1}{2}\text{i}$; **(ii)** $-\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$

N14/32/Q5

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27 The complex number $3 - \text{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

(ii) Showing your working and without using a calculator, express $\frac{u^*}{u}$ in the form $x + \text{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\frac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

Answers: **(i)** Parallelogram **(ii)** $\frac{4}{5} + \frac{3}{5}\text{i}$

N15/32/Q9

HOMEWORK: COMPLEX NUMBERS– VARIANTS 31 & 33

1 The complex number $1 - \mathrm{i}$ is denoted by u.

(i) Showing your working and without using a calculator, express

$$\frac{\mathrm{i}}{u}$$

in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equations $|z - u| = |z|$ and $|z - \mathrm{i}| = 2$. [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part (ii). [3]

33/J15/8

Answers: (i) $-\frac{1}{2}+\frac{1}{2}\mathrm{i}$ (iii) $-\frac{1}{2}\pi$ (270°) and 0.464 radians (26.6°)

2 The complex number w is defined by $w = \dfrac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$.

(i) Without using a calculator, show that $w = 2 + 4\mathrm{i}$. [3]

(ii) It is given that p is a real number such that $\frac{1}{4}\pi \leqslant \arg(w + p) \leqslant \frac{3}{4}\pi$. Find the set of possible values of p. [3]

(iii) The complex conjugate of w is denoted by w^*. The complex numbers w and w^* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through S, T and the origin. [3]

31/J15/8

Answers: (ii) $-6 \le p \le 2$ (iii) $|z - 5| = 5$

3 The complex numbers w and z are defined by $w = 5 + 3\mathrm{i}$ and $z = 4 + \mathrm{i}$.

(i) Express $\dfrac{\mathrm{i}w}{z}$ in the form $x + \mathrm{i}y$, showing all your working and giving the exact values of x and y. [3]

(ii) Find wz and hence, by considering arguments, show that

$$\tan^{-1}\left(\tfrac{3}{5}\right) + \tan^{-1}\left(\tfrac{1}{4}\right) = \tfrac{1}{4}\pi. \qquad [4]$$

33/N14/5

Answer: (i) $-\frac{7}{17}+\frac{23}{17}\mathrm{i}$

4 **Throughout this question the use of a calculator is not permitted.**

The complex numbers w and z satisfy the relation

$$w = \frac{z + \mathrm{i}}{\mathrm{i}z + 2}.$$

(i) Given that $z = 1 + \mathrm{i}$, find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

Answers: (i) $\frac{3}{2}+\frac{1}{2}\mathrm{i}$; (ii) $-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ 31/N14/5

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5 (a) The complex number $\frac{3-5\mathrm{i}}{1+4\mathrm{i}}$ is denoted by u. Showing your working, express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) (i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 - \mathrm{i}| \leqslant 1$ and $|z - \mathrm{i}| \leqslant |z - 2|$. [4]

(ii) Calculate the maximum value of $\arg z$ for points lying in the shaded region. [2]

Answers: (a)(i) $-1 - \mathrm{i}$ (b)(ii) 0.927 radians or 53.1° 33/J14/7

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6 The complex number z is defined by $z = \frac{9\sqrt{3} + 9\mathrm{i}}{\sqrt{3} - \mathrm{i}}$. Find, showing all your working,

(i) an expression for z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

Answers: (i) $9\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$, (ii) $3\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}$, $3\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}$ 31/J14/5

---

7 (a) Without using a calculator, use the formula for the solution of a quadratic equation to solve

$$(2 - \mathrm{i})z^2 + 2z + 2 + \mathrm{i} = 0.$$

Give your answers in the form $a + b\mathrm{i}$. [5]

(b) The complex number w is defined by $w = 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$. In an Argand diagram, the points A, B and C represent the complex numbers w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. [5]

Answer: (i) $-\frac{4}{5}+\frac{3}{5}\mathrm{i}$, $-\mathrm{i}$ (ii) 10; 33/N13/9

8 The complex number z is defined by $z = a + \mathrm{i}b$, where a and b are real. The complex conjugate of z is denoted by z^*.

(i) Show that $|z|^2 = zz^*$ and that $(z - k\mathrm{i})^* = z^* + k\mathrm{i}$, where k is real. [2]

In an Argand diagram a set of points representing complex numbers z is defined by the equation $|z - 10\mathrm{i}| = 2|z - 4\mathrm{i}|$.

(ii) Show, by squaring both sides, that

$$zz^* - 2\mathrm{i}z^* + 2\mathrm{i}z - 12 = 0.$$

Hence show that $|z - 2\mathrm{i}| = 4$. [5]

(iii) Describe the set of points geometrically. [1]

Answer: Circle, centre (0,2), radius 4 33/J13/7

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9 (a) Without using a calculator, solve the equation

$$3w + 2\mathrm{i}w^* = 17 + 8\mathrm{i},$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + b\mathrm{i}$. [4]

(b) In an Argand diagram, the loci

$$\arg(z - 2\mathrm{i}) = \tfrac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3\mathrm{i}|$$

intersect at the point P. Express the complex number represented by P in the form $re^{\mathrm{i}\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

Answer: $7 - 2\mathrm{i}$ *Answer:* $6.69\,e^{\frac{1}{4}\pi\mathrm{i}}$ 31/J13/7

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10 (a) Without using a calculator, solve the equation $\mathrm{i}w^2 = (2 - 2\mathrm{i})^2$. [3]

(b) (i) Sketch an Argand diagram showing the region R consisting of points representing the complex numbers z where

$$|z - 4 - 4\mathrm{i}| \leqslant 2. \quad [2]$$

(ii) For the complex numbers represented by points in the region R, it is given that

$$p \leqslant |z| \leqslant q \quad \text{and} \quad \alpha \leqslant \arg z \leqslant \beta.$$

Find the values of p, q, α and β, giving your answers correct to 3 significant figures. [6]

Answers: **(a)** $\pm i\sqrt{8}$; **(b) (ii)** 3.66, 7.66, 0.424, 1.15. 33/N12/10

11 The complex number $1+(\sqrt{2})\mathrm{i}$ is denoted by u. The polynomial x^4+x^2+2x+6 is denoted by $\mathrm{p}(x)$.

(i) Showing your working, verify that u is a root of the equation $\mathrm{p}(x)=0$, and write down a second complex root of the equation. [4]

(ii) Find the other two roots of the equation $\mathrm{p}(x)=0$. [6]

Answers: (i) $1-\sqrt{2}\ \mathrm{i}$; (ii) $-1+\mathrm{i}, -1-\mathrm{i}$. 31/N12/9

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12 (a) The complex numbers u and w satisfy the equations

$$u-w=4\mathrm{i} \quad \text{and} \quad uw=5.$$

Solve the equations for u and w, giving all answers in the form $x+\mathrm{i}y$, where x and y are real. [5]

(b) (i) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z-2+2\mathrm{i}| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of z. [5]

(ii) Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

Answers: (a) $u=1+2\mathrm{i}$ and $w=1-2\mathrm{i}$; $u=-1+2\mathrm{i}$ and $w=-1-2\mathrm{i}$ (b) (ii) $2+\sqrt{2}$ 33/J12/10

---

13 The complex number u is defined by $u=\dfrac{(1+2\mathrm{i})^2}{2+\mathrm{i}}$.

(i) Without using a calculator and showing your working, express u in the form $x+\mathrm{i}y$, where x and y are real. [4]

(ii) Sketch an Argand diagram showing the locus of the complex number z such that $|z-u|=|u|$. [3]

31/J12/4

Answer: (i) $u=\dfrac{-2}{5}+\dfrac{11}{5}\mathrm{i}$

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14 The complex number w is defined by $w=-1+\mathrm{i}$.

(i) Find the modulus and argument of w^2 and w^3, showing your working. [4]

(ii) The points in an Argand diagram representing w and w^2 are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z-(a+b\mathrm{i})|=k$. [4]

33/N11/6

Answers: (i) $2,\ 2\sqrt{2},\ -\dfrac{1}{2}\pi,\ \dfrac{1}{4}\pi$; (ii) $\left|z+\dfrac{1}{2}+\dfrac{1}{2}i\right|=\dfrac{1}{2}\sqrt{10}$

15 (a) Showing your working, find the two square roots of the complex number $1-(2\sqrt{6})\text{i}$. Give your answers in the form $x+\text{i}y$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z-3\text{i}| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

Answer: (i) $\pm(\sqrt{3}-\text{i}\sqrt{2})$ (ii) 131.8° (or 2.30 radians) 31/N11/10

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16 (i) Find the roots of the equation

$$z^2+(2\sqrt{3})z+4=0,$$

giving your answers in the form $x+\text{i}y$, where x and y are real. [2]

(ii) State the modulus and argument of each root. [3]

(iii) Showing all your working, verify that each root also satisfies the equation

$$z^6=-64.$$ [3]

Answers: (i) $-\sqrt{3}+\text{i}$, $-\sqrt{3}-\text{i}$; (ii) 2, $\frac{5}{6}\pi$; 2, $-\frac{5}{6}\pi$. 33/J11/7

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17 The complex number u is defined by $u=\dfrac{6-3\text{i}}{1+2\text{i}}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z-u)=\frac{1}{4}\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z-(1+\text{i})u|=1$, find the greatest possible value of $|z|$. [3]

Answers: (i) 3; (ii) $\frac{3}{2}\sqrt{2}$ or 2.12; (iii) $3\sqrt{2}+1$ or 5.24. 31/J11/8

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18 The complex number w is defined by $w=2+\text{i}$.

(i) Showing your working, express w^2 in the form $x+\text{i}y$, where x and y are real. Find the modulus of w^2. [3]

(ii) Shade on an Argand diagram the region whose points represent the complex numbers z which satisfy

$$|z-w^2| \leqslant |w^2|.$$ [3]

Answers: (i) $3+4\text{i}$, 5. 33/N10/3

19 The polynomial $p(z)$ is defined by

$$p(z) = z^3 + mz^2 + 24z + 32,$$

where m is a constant. It is given that $(z + 2)$ is a factor of $p(z)$.

(i) Find the value of m. [2]

(ii) Hence, showing all your working, find

(a) the three roots of the equation $p(z) = 0$, [5]

(b) the six roots of the equation $p(z^2) = 0$. [6]

Answers: **(i)** 6; **(ii)(a)** -2, $-2 \pm 2\sqrt{3}\mathrm{i}$, **(b)** $\pm \mathrm{i}\sqrt{2}$, $\pm(1+\mathrm{i}\sqrt{3})$, $\pm(1-\mathrm{i}\sqrt{3})$. **33/N10/10**

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20 The complex number z is given by

$$z = (\sqrt{3}) + \mathrm{i}.$$

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z^*. Showing your working, express in the form $x + \mathrm{i}y$, where x and y are real,

(a) $2z + z^*$,

(b) $\dfrac{\mathrm{i}z^*}{z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and $\mathrm{i}z^*$ respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

Answers: **(i)** 2, $\frac{1}{6}\pi$; **(ii)(a)** $3\sqrt{3} + \mathrm{i}$, **(b)** $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$. **31/N10/6**

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21 (a) The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + \mathrm{i}\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + \mathrm{i}\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1 - \mathrm{i}\sqrt{3}| \leqslant 1$ and $\arg z \leqslant \frac{1}{3}\pi$. [5]

Answer: **(a)** $1 - \mathrm{i}\sqrt{3}$. **33/J10/8**

22 The complex number $2 + 2i$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - i|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

Answers: **(i)** $\sqrt{8}$, 45°; **(iii)** $\sqrt{7}$. **31/J10/7**

23 **(a)** It is given that $(1 + 3i)w = 2 + 4i$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

(b) On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $re^{i\theta}$. [4]

Answer: **(a)** -0.284 radians, **(b)** $5e^{\pm i\frac{\pi}{3}}$ **N15/33/Q9**

24 **Throughout this question the use of a calculator is not permitted.**

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})i$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + iy$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{iz}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and iz respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

Answers: Modulus $2\sqrt{2}$ Argument $-\pi/3$ or $-60°$ **N16/33/Q7**

Answer: $3\sqrt{2} + \sqrt{6}i$ ½ $\sqrt{3}$ + ½ i

25 **(a)** Showing all necessary working, express the complex number $\dfrac{2+3i}{1-2i}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the values of r and θ correct to 3 significant figures. [5]

(b) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - 3 + 2i| = 1$. Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [4]

Answers: **(a)** $1.61e^{i2.09}$ **(b)** $\sqrt{13} - 1$ **N18/33/Q8**

26 **(a)** Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) **(i)** On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2\mathrm{i}| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

Answers: **(a)** $\pm\left(3 - \mathrm{i}\sqrt{2}\right)$, **(b)(ii)** $\sqrt{2} - 1$ **J16/31/Q10**

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27 **Throughout this question the use of a calculator is not permitted.**

The complex numbers $-1 + 3\mathrm{i}$ and $2 - \mathrm{i}$ are denoted by u and v respectively. In an Argand diagram with origin O, the points A, B and C represent the numbers u, v and $u + v$ respectively.

(i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]

(ii) Find, in the form $x + \mathrm{i}y$, where x and y are real, the complex number $\dfrac{u}{v}$. [3]

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

J16/33/Q9

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28 **Throughout this question the use of a calculator is not permitted.**

The complex numbers u and w are defined by $u = -1 + 7\mathrm{i}$ and $w = 3 + 4\mathrm{i}$.

(i) Showing all your working, find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - 2w$ and $\dfrac{u}{w}$. [4]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, w and $u - 2w$ respectively.

(ii) Prove that angle $AOB = \frac{1}{4}\pi$. [2]

(iii) State fully the geometrical relation between the line segments OB and CA. [2]

Answers: **(i)** $-7 - \mathrm{i}$, $1 + \mathrm{i}$ **(iii)** parallel, $|CA| = 2|OB|$ **J17/31/Q7**

29 **Throughout this question the use of a calculator is not permitted.**

(a) The complex numbers z and w satisfy the equations

$$z + (1 + \mathrm{i})w = \mathrm{i} \qquad \text{and} \qquad (1 - \mathrm{i})z + \mathrm{i}w = 1.$$

Solve the equations for z and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex numbers u and v are given by $u = 1 + (2\sqrt{3})\mathrm{i}$ and $v = 3 + 2\mathrm{i}$. In an Argand diagram, u and v are represented by the points A and B. A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90°$. Find the complex number z represented by C, giving your answer in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

Answers: **(a)** $w = -\frac{1}{5} + \frac{2}{5}\mathrm{i}$, $z = \frac{3}{5} + \frac{4}{5}\mathrm{i}$ **(b)** $4\sqrt{3} - 1 + 6\mathrm{i}$ **J17/33/Q11**

---

30 **(i)** Showing all working and without using a calculator, solve the equation $z^2 + (2\sqrt{6})z + 8 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) The points representing the roots are A and B, and O is the origin. Find angle AOB. [3]

(iv) Prove that triangle AOB is equilateral. [1]

Answer: **(i)** $z = -\sqrt{6} \pm \sqrt{2}i$ **(iii)** 60° **J18/31/Q7**

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31 **(a)** Find the complex number z satisfying the equation

$$3z - \mathrm{i}z^* = 1 + 5\mathrm{i},$$

where z^* denotes the complex conjugate of z. [4]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| \leqslant 3$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. Calculate the greatest value of $\arg z$ for points in this region. Give your answer in radians correct to 2 decimal places. [5]

Answers: **(a)** $1 + 2\mathrm{i}$ **(b)** 2.41 **J18/33/Q9**

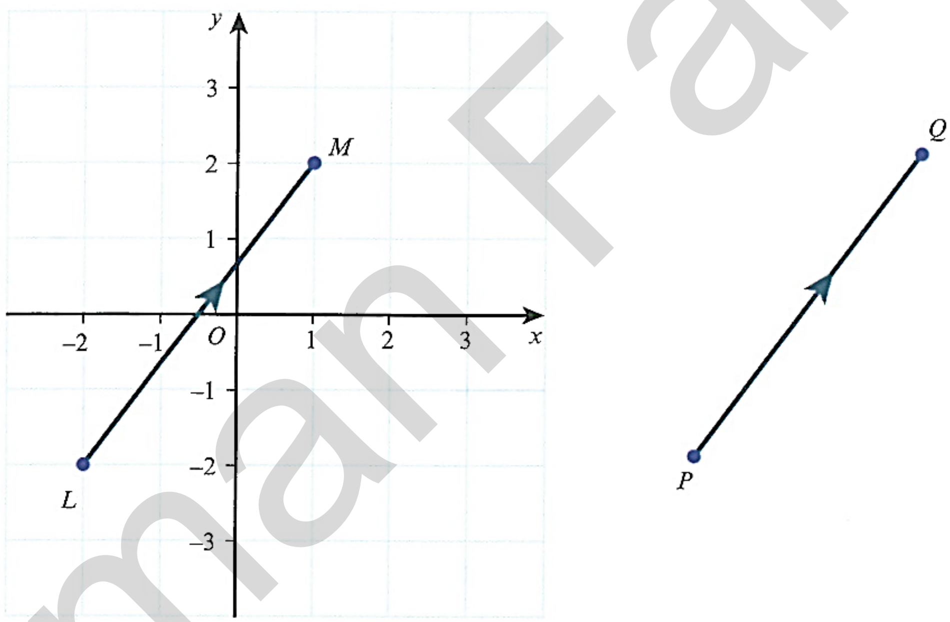
CHAPTER 20: VECTORS

Recall that a vector has both **magnitude** (size) and **direction**.

A movement through a certain distance and in a given direction is called a **displacement** or a **translation**. The magnitude or length of a displacement represents the distance moved.

Examples of displacement or translation vectors

- Sami walks 100 m in a straight line on a bearing of 045° from place A to place B. Jan is also standing at place A. Jan walks directly to Sami. Jan therefore also walks for 100 m on a bearing of 045° from place A. Jan follows the same displacement vector as Sami.
- The diagram shows two arrows.



The arrow from L to M is the same length and has been drawn in the same direction as the arrow from P to Q.

This means that the points L and P have been displaced or translated by the same vector to M and Q, respectively.

The arrow LM and the arrow PQ represent the same displacement.

Example 1

Example 6.1 *Given that* $\mathrm{a} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$, $\mathrm{b} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}$ *and* $\mathrm{c} = \begin{pmatrix} 3 \\ -5 \\ -2 \end{pmatrix}$, *evaluate*

(i) $\mathrm{a}+\mathrm{b}$ (ii) $\mathrm{c}-\mathrm{b}$ (iii) $\mathrm{b}+\mathrm{a}-\mathrm{c}$

Example 2

Example 6.2 *Using the same vectors as in Example 6.1, evaluate*
(i) $2\mathrm{a}+3\mathrm{b}$ (ii) $5\mathrm{b}+3\mathrm{c}-4\mathrm{a}$

Parallel Vectors

Example 3

Example 6.4 *Given that* $\mathrm{a} = \begin{pmatrix} 2 \\ p \\ -5 \end{pmatrix}$ *and* $\mathrm{b} = \begin{pmatrix} -6 \\ 9 \\ q \end{pmatrix}$ *are parallel, find the values of the constants* p *and* q.

Dot Product of Vectors

Example 4

Example 6.5 *Given that* $\mathrm{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ *and* $\mathrm{b} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, *evaluate the dot product* $\mathrm{a} \cdot \mathrm{b}$.

Perpendicular Vectors

Example 5

Example 6.6 *Given that* $\mathrm{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ *and* $\mathrm{b} = \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}$, *show that* a *and* b *are perpendicular.*

Magnitude of a Vector

Example 6

Given that $\mathrm{a} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, *find the length of* a.

Unit Vector

Example 7

Example 6.8 *Given that* $\mathrm{a} = 2\mathrm{i} + \mathrm{j} + 2\mathrm{k}$, *find a unit vector and a vector of length 9 in the direction of* a.

Angles between Vectors

Example 8

Find the angle between $\mathrm{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ *and* $\mathrm{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$.

Example 9

Relative to the origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}.$$

Find angle OBA.

Position vector of a point between two points

Example 10

Example 6.10 *The position vectors of A, B and C are given by* $\overrightarrow{OA} = 2\mathrm{i} + 3\mathrm{j} - 4\mathrm{k}$, $\overrightarrow{OB} = 5\mathrm{i} - \mathrm{j} + 2\mathrm{k}$ *and* $\overrightarrow{OC} = p\mathrm{i} - 9\mathrm{j} + 14\mathrm{k}$. *Find*

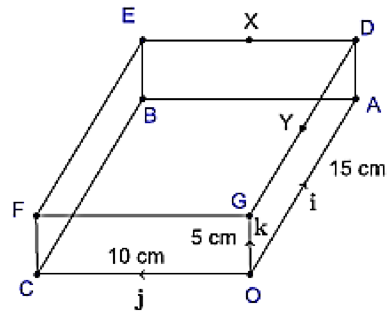
(i) *the position vector of D such that ABCD is a parallelogram,*

(ii) *the value of* p *if A, B and C are collinear,*

(iii) *the position vector of Q if* $AQ : QB = 3 : 4$.

Example 11

Example 6.11 *Consider the cuboid OABCDEFG. The unit vectors* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are along* $\overrightarrow{OA}$, $\overrightarrow{OC}$ *and* $\overrightarrow{OG}$ *respectively. Also* $OC = 10$ *cm,* $OA = 15$ *cm and* $OG = 5$ *cm and* X *and* Y *are the midpoint of* DE *and* GD *respectively.*



Find

(i) $\overrightarrow{OX}$ *in terms of* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$,

(ii) $\overrightarrow{OY}$ *in terms of* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$,

(iii) $X\widehat{O}Y$.

Example 12

Example 6.12 *Given that* $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$, $\overrightarrow{OR} = \begin{pmatrix} 2 \\ 3 \\ x \end{pmatrix}$ *and* $\overrightarrow{OS} = \begin{pmatrix} -2 \\ 0 \\ y \end{pmatrix}$, *find*

(i) *a unit vector in the direction of* $\overrightarrow{PQ}$,

(ii) *the value of* x *such that* $P\widehat{O}R = 90^0$,

(iii) *the values of* y *for which* $|\overrightarrow{PS}| = 6$ *units.*

Example 13

Example 6.13 *Given that* $\overrightarrow{OP} = \mathrm{i} + 2\mathrm{j} + 4\mathrm{k}$ *and* $\overrightarrow{OQ} = 4\mathrm{i} - 2\mathrm{j} + 4\mathrm{k}$,

(i) *Calculate* $Q\widehat{P}O$.

(ii) *Find a vector of length 25 in the direction* $\overrightarrow{PQ}$.

(iii) *Given also that* $\overrightarrow{OR} = 3\mathrm{i} + x\mathrm{k}$, *where* x *is a constant and* $PR = 2PQ$, *find the possible values of* x.

Example 14

June 2014/11 Question 8

Relative to an origin O, the position vectors of points A and B are given by

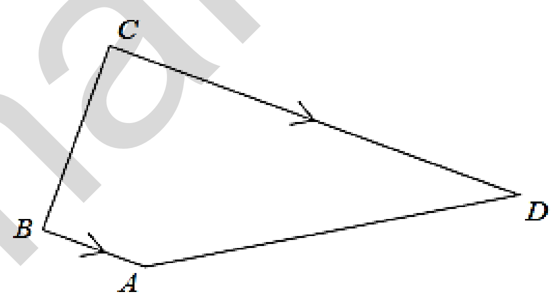
$$\overrightarrow{OA} = \begin{pmatrix} 3p \\ 4 \\ p^2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} -p \\ -1 \\ p^2 \end{pmatrix}.$$

(i) Find the values of p for which angle AOB is $90°$. [3]

(ii) For the case where $p = 3$, find the unit vector in the direction of $\overrightarrow{BA}$. [3]

Example 15

June 2014/12 Question 7



The diagram shows a trapezium $ABCD$ in which BA is parallel to CD. The position vectors of A, B and C relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

(i) Use a scalar product to show that AB is perpendicular to BC. [3]

(ii) Given that the length of CD is 12 units, find the position vector of D. [4]

Example 16

June 2014/13 Question 7

The position vectors of points A, B and C relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$$

(i) Show that angle $BAC = \cos^{-1}\left(\frac{1}{3}\right)$. [5]

(ii) Use the result in part (i) to find the exact value of the area of triangle ABC. [3]

Example 17

November 2014/11 Question 6

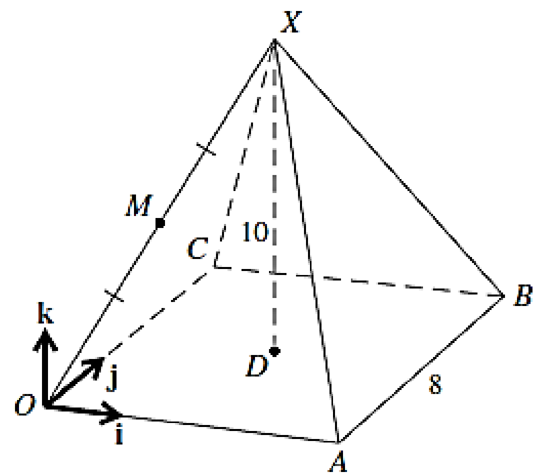
Relative to an origin O, the position vector of A is $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and the position vector of B is $7\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

(i) Show that angle OAB is a right angle. [4]

(ii) Find the area of triangle OAB. [3]

Example 18

November 2014/12 Question 7



The diagram shows a pyramid $OABCX$. The horizontal square base $OABC$ has side 8 units and the centre of the base is D. The top of the pyramid, X, is vertically above D and $XD = 10$ units. The mid-point of OX is M. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively and the unit vector $\mathbf{k}$ is vertically upwards.

(i) Express the vectors $\overrightarrow{AM}$ and $\overrightarrow{AC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(ii) Use a scalar product to find angle MAC. [4]

Example 19

November 2014/13 Question 7

Three points, O, A and B, are such that $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + p\mathbf{k}$ and $\overrightarrow{OB} = -7\mathbf{i} + (1-p)\mathbf{j} + p\mathbf{k}$, where p is a constant.

(i) Find the values of p for which $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$. [3]

(ii) The magnitudes of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are a and b respectively. Find the value of p for which $b^2 = 2a^2$. [2]

(iii) Find the unit vector in the direction of $\overrightarrow{AB}$ when $p = -8$. [3]

Example 20

June 2014/32 Question 10

Referred to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

(i) Find the exact value of the cosine of angle BAC. [4]

(ii) Hence find the exact value of the area of triangle ABC. [3]

Example 21

June 2013/11 Question 6

Relative to an origin O, the position vectors of three points, A, B and C, are given by

$$\overrightarrow{OA} = \mathbf{i} + 2p\mathbf{j} + q\mathbf{k}, \quad \overrightarrow{OB} = q\mathbf{j} - 2p\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = -(4p^2 + q^2)\mathbf{i} + 2p\mathbf{j} + q\mathbf{k},$$

where p and q are constants.

(i) Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$ for all non-zero values of p and q. [2]

(ii) Find the magnitude of $\overrightarrow{CA}$ in terms of p and q. [2]

(iii) For the case where $p = 3$ and $q = 2$, find the unit vector parallel to $\overrightarrow{BA}$. [3]

Example 22

June 2013/12 Question 6

Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 3\mathbf{i} + p\mathbf{j} + q\mathbf{k},$$

where p and q are constants.

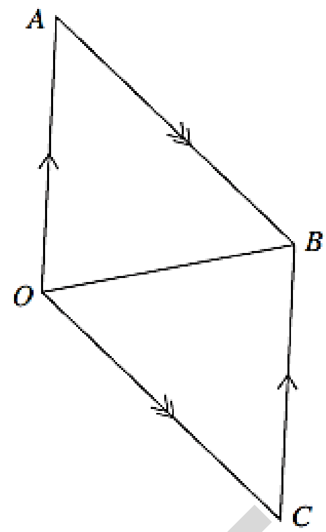
(i) State the values of p and q for which $\overrightarrow{OA}$ is parallel to $\overrightarrow{OB}$. [2]

(ii) In the case where $q = 2p$, find the value of p for which angle BOA is 90°. [2]

(iii) In the case where $p = 1$ and $q = 8$, find the unit vector in the direction of $\overrightarrow{AB}$. [3]

Example 23

June 2013/13 Question 8



The diagram shows a parallelogram $OABC$ in which

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}.$$

(i) Use a scalar product to find angle BOC. [6]

(ii) Find a vector which has magnitude 35 and is parallel to the vector $\overrightarrow{OC}$. [2]

Vector Equation of a line

Example 24

a) Write down a vector equation for the straight line through the point (3, 1) with direction vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

b) Find a vector equation for the line through the points $L(1, 2)$ and $M(5, 3)$.

Example 25

Points A and B have position vectors $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$ respectively.

a) Find, in vector form, an equation of the straight line that passes through A and B.

b) Show that the point $C(-12, 13, 8)$ lies on this line.

Example 26

Example 16.1 *Find the equation vector form for the following*

(a) *line passing through the point with position vector* $\mathrm{i} + \mathrm{j} + \mathrm{k}$ *and parallel to* $2\mathrm{i} - \mathrm{j} - 2\mathrm{k}$.

(b) a line passing through a point $A(1, 2, 3)$ *and point* $B(-1, 3, 2)$. **S**

Example 27

Example 16.1 *Find the equation vector form for the following*

(a) *line passing through the point with position vector* $\mathrm{i} + \mathrm{j} + \mathrm{k}$ *and parallel to* $2\mathrm{i} - \mathrm{j} - 2\mathrm{k}$.

(b) a line passing through a point $A(1, 2, 3)$ *and point* $B(-1, 3, 2)$. **S**

Example 28

Example 16.2 *Show that the point with the position vector* $\mathrm{i} + 3\mathrm{j} - 2\mathrm{k}$ *lies on the line with vector equation* $\mathrm{r} = 5\mathrm{i} + \mathrm{j} + 2\mathrm{k} + \lambda\,(2\mathrm{i} - \mathrm{j} + 2\mathrm{k})$.

Parametric Form

$R(x, y, z)$ represents any point on the line.

The position vector of this point can be written as $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

In the equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, using the position vectors in component form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ or } x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + t(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

Simplifying the right-hand side of this:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix} \text{ or } \begin{array}{l} x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a_1\mathbf{i} + tb_1\mathbf{i} + a_2\mathbf{j} + tb_2\mathbf{j} + a_3\mathbf{k} + tb_3\mathbf{k} \\ x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a_1 + tb_1)\mathbf{i} + (a_2 + tb_2)\mathbf{j} + (a_3 + tb_3)\mathbf{k} \end{array}$$

Comparing the components:

$x = a_1 + tb_1$

$y = a_2 + tb_2$

$z = a_3 + tb_3$

These three equations are called the **parametric equations** of the line.

Parallel Lines

Example 29

Example 16.3 *Show that the lines*

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad L_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + s\begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix}$$

are parallel.

Intersection of two lines

Example 30

Example 16.4 *Find the point of intersection of the lines*

$$\mathrm{r} = \mathrm{i} + \mathrm{k} + \lambda\,(\mathrm{i} - 2\mathrm{j} - 3\mathrm{k}) \text{ and } \mathrm{r} = 3\mathrm{i} - 5\mathrm{k} + \mu\,(-\mathrm{i} + 3\mathrm{j} + 3\mathrm{k}).$$

Example 31

Example 16.6 *The lines with their respective vector equations*

$$\mathrm{r} = a\mathrm{i} + \mathrm{j} + \mathrm{k} + \lambda\,(3\mathrm{i} - 2\mathrm{j} + \mathrm{k}) \text{ and } \mathrm{r} = -2\mathrm{i} + 3\mathrm{j} + \mu\,(2\mathrm{i} + 3\mathrm{j} - \mathrm{k})$$

intersect , find the value of a and the position vector of the point of intersection.

Skew Lines

Example 32

Example 16.5 *Determine whether the lines with their respective vector equations*

$$\mathrm{r} = \mathrm{i} + \mathrm{k} + \lambda\,(-\mathrm{i} + 3\mathrm{j} + 3\mathrm{k}) \text{ and } \mathrm{r} = 2\mathrm{i} + 3\mathrm{j} + \mu\,(\mathrm{i} - \mathrm{j} + \mathrm{k})$$

intersect or not.

Angle between two lines

Example 33

Example 16.7 *Find the acute angle between the lines*

$$\mathrm{r} = 4\mathrm{i} - \mathrm{j} + \mathrm{k} + \lambda\,(\mathrm{i} + 2\mathrm{k}) \text{ and } \mathrm{r} = \mathrm{i} - \mathrm{j} + 2\mathrm{k} + \mu\,(3\mathrm{i} + 2\mathrm{j} - \mathrm{k}).$$

Perpendicular distance from a point to a line

Example 34

Example 16.8 *Find the perpendicular distance from the point with position vector* $3\mathrm{i} - \mathrm{k}$ *to the line* $\mathrm{r} = \mathrm{i} - 2\mathrm{j} + \mathrm{k} + \lambda\,(3\mathrm{i} + 4\mathrm{j} + 5\mathrm{k})$

Example 34

November 2014/31 Question 10

The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

Example 35

November 2014/33 Question 7

The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = a\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3a\mathbf{k}),$$

where a is a constant.

(i) Show that the lines intersect for all values of a. [4]

(ii) Given that the point of intersection is at a distance of 9 units from the origin, find the possible values of a. [4]

HOMEWORK: VECTORS VARIANT 32

1

The points A, B, C and D have position vectors $3\mathbf{i} + 2\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $2\mathbf{j} + 7\mathbf{k}$ and $-2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$ respectively.

(i) Use a scalar product to show that BA and BC are perpendicular. [4]

(ii) Show that BC and AD are parallel and find the ratio of the length of BC to the length of AD. [4]

Answers: (i) Proof; (ii) Proof, 2:5. **J03/Q8**

2

Relative to an origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ 2 \\ p \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} -1 \\ 0 \\ q \end{pmatrix},$$

where p and q are constants. Find

(i) the unit vector in the direction of $\overrightarrow{AB}$, [3]

(ii) the value of p for which angle $AOC = 90°$, [3]

(iii) the values of q for which the length of $\overrightarrow{AD}$ is 7 units. [4]

Answers: (i) $\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$; (ii) 10; (iii) 5 or −7. **J04/Q9**

3

Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

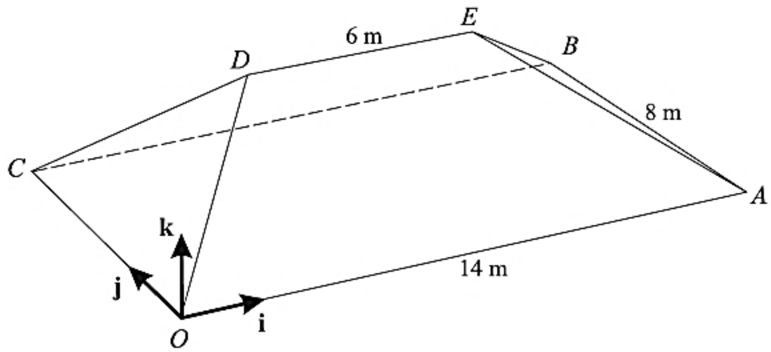
(i) Use a scalar product to find angle AOB, correct to the nearest degree. [4]

(ii) Find the unit vector in the direction of $\overrightarrow{AB}$. [3]

(iii) The point C is such that $\overrightarrow{OC} = 6\mathbf{j} + p\mathbf{k}$, where p is a constant. Given that the lengths of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal, find the possible values of p. [4]

Answers: (i) 99°; (ii) $\frac{1}{7}(2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k})$; (iii) p = −7 or 5. **J05/Q11**

4



The diagram shows the roof of a house. The base of the roof, $OABC$, is rectangular and horizontal with $OA = CB = 14$ m and $OC = AB = 8$ m. The top of the roof DE is 5 m above the base and $DE = 6$ m. The sloping edges OD, CD, AE and BE are all equal in length.

Unit vectors **i** and **j** are parallel to OA and OC respectively and the unit vector **k** is vertically upwards.

(i) Express the vector $\overrightarrow{OD}$ in terms of **i**, **j** and **k**, and find its magnitude. [4]

(ii) Use a scalar product to find angle DOB. [4]

Answers: (i) 4i + 4j + 5k, 7.55 m; (ii) 43.7° (or 0.763 radians). J06/Q8

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5 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}.$$

(i) Given that C is the point such that $\overrightarrow{AC} = 2\overrightarrow{AB}$, find the unit vector in the direction of $\overrightarrow{OC}$. [4]

The position vector of the point D is given by $\overrightarrow{OD} = \begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix}$, where k is a constant, and it is given that $\overrightarrow{OD} = m\overrightarrow{OA} + n\overrightarrow{OB}$, where m and n are constants.

(ii) Find the values of m, n and k. [4]

J07/Q9

Answers: (i) $\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$; (ii) $m = -2$, $n = 3$, $k = -8$.

6 Relative to an origin O, the position vectors of points A and B are $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + p\mathbf{k}$ respectively.

(i) Find the value of p for which OA and OB are perpendicular. [2]

(ii) In the case where $p = 6$, use a scalar product to find angle AOB, correct to the nearest degree. [3]

(iii) Express the vector $\overrightarrow{AB}$ is terms of p and hence find the values of p for which the length of AB is 3.5 units. [4]

Answers: (i) -2; (ii) $40°$; (iii) 0.5 or 3.5. **J08/Q10**

7 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

(i) Find the value of $\overrightarrow{OA}.\overrightarrow{OB}$ and hence state whether angle AOB is acute, obtuse or a right angle. [3]

(ii) The point X is such that $\overrightarrow{AX} = \frac{2}{5}\overrightarrow{AB}$. Find the unit vector in the direction of OX. [4]

Answers: (i) -6, obtuse; (ii) $⅓(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **J09/Q6**

8 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 1 \\ p \end{pmatrix}.$$

(i) Find the value of p for which $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$. [2]

(ii) Find the values of p for which the magnitude of $\overrightarrow{AB}$ is 7. [4]

Answers: (i) 5; (ii) 4, -2. **J10/12/Q5**

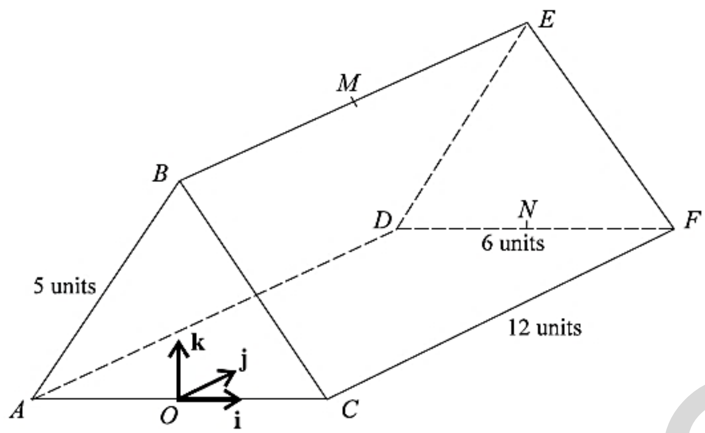
9 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} p \\ p \\ p+1 \end{pmatrix}$, find

(i) the angle between the directions of **a** and **b**, [4]

(ii) the value of p for which **b** and **c** are perpendicular. [3]

Answers: (i)103.8°; (ii) $-\frac{3}{11}$. **N02/Q7**

10



The diagram shows a triangular prism with a horizontal rectangular base $ADFC$, where $CF = 12$ units and $DF = 6$ units. The vertical ends ABC and DEF are isosceles triangles with $AB = BC = 5$ units. The mid-points of BE and DF are M and N respectively. The origin O is at the mid-point of AC.

Unit vectors **i**, **j** and **k** are parallel to OC, ON and OB respectively.

(i) Find the length of OB. [1]

(ii) Express each of the vectors $\overrightarrow{MC}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(iii) Evaluate $\overrightarrow{MC}.\overrightarrow{MN}$ and hence find angle CMN, giving your answer correct to the nearest degree. [4]

Answers: (i) 4 units; (ii) $\overrightarrow{MC} = 3\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$, $\overrightarrow{MN} = 6\mathbf{j} - 4\mathbf{k}$; (iii) −20, 111°. N03/Q7

11 The points A and B have position vectors $\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$ and $-5\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ respectively, relative to an origin O.

(i) Use a scalar product to calculate angle AOB, giving your answer in radians correct to 3 significant figures. [4]

(ii) The point C is such that $\overrightarrow{AB} = 2\overrightarrow{BC}$. Find the unit vector in the direction of $\overrightarrow{OC}$. [4]

Answers: (i) 0.907 radians; (ii) $\frac{1}{12}(-8\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$. N04/Q8

12 Relative to an origin O, the position vectors of points P and Q are given by

$$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ q \end{pmatrix},$$

where q is a constant.

(i) In the case where $q = 3$, use a scalar product to show that $\cos POQ = \frac{1}{7}$. [3]

(ii) Find the values of q for which the length of $\overrightarrow{PQ}$ is 6 units. [4]

Answer: (ii) $q = 5$ or -3. N05/Q4

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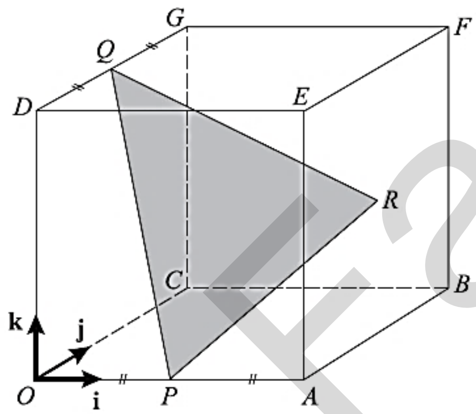
13 The position vectors of points A and B are $\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ respectively, relative to an origin O.

(i) Calculate angle AOB. [3]

(ii) The point C is such that $\overrightarrow{AC} = 3\overrightarrow{AB}$. Find the unit vector in the direction of $\overrightarrow{OC}$. [4]

Answer: (i) 36.7°; (ii) $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$. N06/Q4

14



The diagram shows a cube $OABCDEFG$ in which the length of each side is 4 units. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The mid-points of OA and DG are P and Q respectively and R is the centre of the square face $ABFE$.

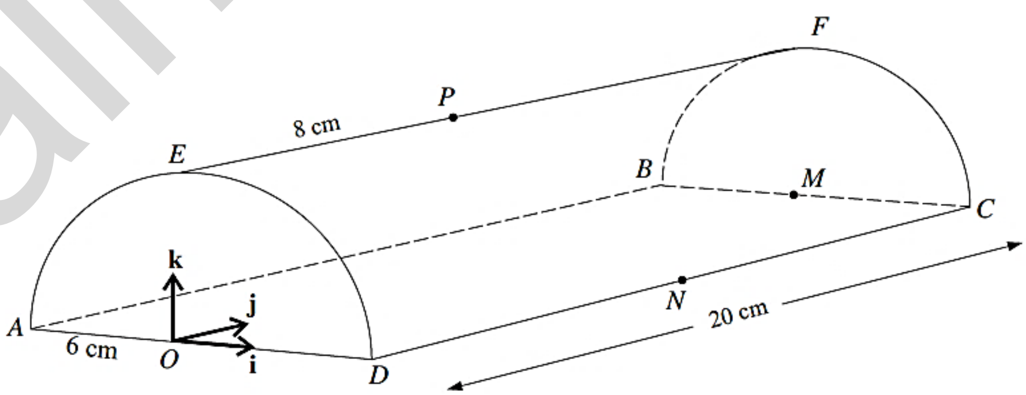
(i) Express each of the vectors $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(ii) Use a scalar product to find angle QPR. [4]

(iii) Find the perimeter of triangle PQR, giving your answer correct to 1 decimal place. [3]

Answers: (i) $2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $-2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$; (ii) 61.9°; (iii) 12.8 cm. N07/Q10

15



The diagram shows a semicircular prism with a horizontal rectangular base $ABCD$. The vertical ends AED and BFC are semicircles of radius 6 cm. The length of the prism is 20 cm. The mid-point of AD is the origin O, the mid-point of BC is M and the mid-point of DC is N. The points E and F are the highest points of the semicircular ends of the prism. The point P lies on EF such that $EP = 8$ cm.

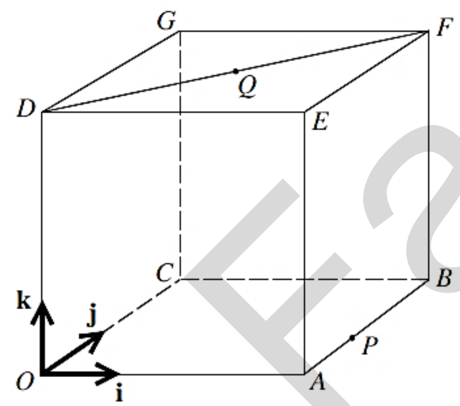
Unit vectors **i**, **j** and **k** are parallel to OD, OM and OE respectively.

(i) Express each of the vectors $\overrightarrow{PA}$ and $\overrightarrow{PN}$ in terms of **i**, **j** and **k**. [3]

(ii) Use a scalar product to calculate angle APN. [4]

Answers: (i) $-6\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$, $6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$; (ii) 99.0°. N08/Q4

16



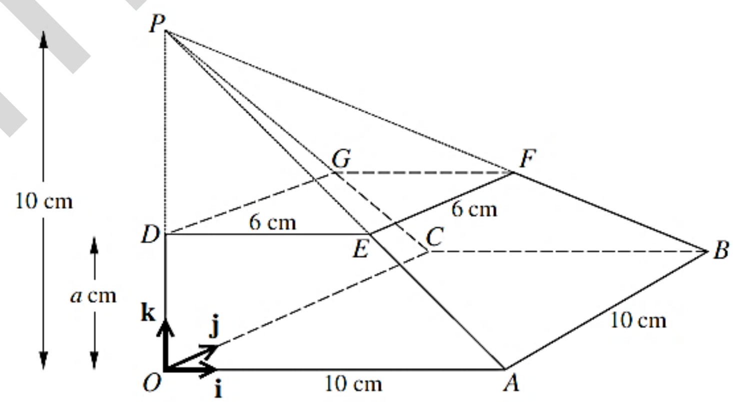
In the diagram, $OABCDEFG$ is a cube in which each side has length 6. Unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The point P is such that $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$ and the point Q is the mid-point of DF.

(i) Express each of the vectors $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$ in terms of **i**, **j** and **k**. [3]

(ii) Find the angle OQP. [4]

Answers: (i) $3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, $-3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$; (ii) 53.0°. N09/12/Q6

17



The diagram shows a pyramid $OABCP$ in which the horizontal base $OABC$ is a square of side 10 cm and the vertex P is 10 cm vertically above O. The points D, E, F, G lie on OP, AP, BP, CP respectively and $DEFG$ is a horizontal square of side 6 cm. The height of $DEFG$ above the base is a cm. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively.

(i) Show that $a = 4$. [2]

(ii) Express the vector $\overrightarrow{BG}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [2]

(iii) Use a scalar product to find angle GBA. [4]

Answers: (ii) $-10\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$; (iii) $69.6°$. N10/12/Q9

18 The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect, and find the position vector of their point of intersection. [5]

Answers: (i) $3\mathbf{i} + \mathbf{j} + \mathbf{k}$; (ii) $7x + y - 5z = 17$. N03/Q10

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19 The lines l and m have vector equations

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m do not intersect. [4]

The point P lies on l and the point Q has position vector $2\mathbf{i} - \mathbf{k}$.

(ii) Given that the line PQ is perpendicular to l, find the position vector of P. [4]

(iii) Verify that Q lies on m and that PQ is perpendicular to m. [2]

Answer: (ii) $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. N04/Q9

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20 With respect to the origin O, the points A and B have position vectors given by

$$\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \overrightarrow{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Prove that the line l does not intersect the line through A and B. [5]

Answer: (ii) $6x + y - 8z = 6$. J05/Q10

21 The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}.$$

The line l passes through A and is parallel to OB. The point N is the foot of the perpendicular from B to l.

(i) State a vector equation for the line l. [1]

(ii) Find the position vector of N and show that $BN = 3$. [6]

J06/Q10

Answers: (i) $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$; (ii) $\begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$; (iii) $7x - 11y + 8z = 0$.

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22 The points A and B have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation

$$\mathbf{r} = (1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}.$$

(i) Show that l does not intersect the line passing through A and B. [4]

(ii) The point P lies on l and is such that angle PAB is equal to $60°$. Given that the position vector of P is $(1 - 2t)\mathbf{i} + (5 + t)\mathbf{j} + (2 - t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P. [6]

Answer: (ii) $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. J08/Q10

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23 The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. It is given that l lies in the plane with equation $2x + by + cz = 1$, where b and c are constants.

(i) Find the values of b and c. [6]

(ii) The point P has position vector $2\mathbf{j} + 4\mathbf{k}$. Show that the perpendicular distance from P to l is $\sqrt{5}$. [5]

Answers: (i) -2, 3. J09/Q9

24 With respect to the origin O, the points A, B, C, D have position vectors given by

$$\overrightarrow{OA} = 4\mathbf{i} + \mathbf{k}, \quad \overrightarrow{OB} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j}, \quad \overrightarrow{OD} = -\mathbf{i} - 4\mathbf{k}.$$

(i) Calculate the acute angle between the lines AB and CD. [4]

(ii) Prove that the lines AB and CD intersect. [4]

(iii) The point P has position vector $\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. Show that the perpendicular distance from P to the line AB is equal to $\sqrt{3}$. [4]

Answer: (i) 45.6°. **N02/Q10**

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25 With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line AB and OP is perpendicular to AB.

(i) Find a vector equation for the line AB. [1]

(ii) Find the position vector of P. [4]

Answers: **(i)** $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$; (ii) $\frac{2}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$; (iii) $2x + 5y + 7z = 26$. **N10/32/Q7**

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26 Referred to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = 3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

(i) Find the exact value of the cosine of angle BAC. [4

(ii) Hence find the exact value of the area of triangle ABC. [3

Answer: $\frac{20}{21}$; **(ii)** $\frac{\sqrt{41}}{2}$; **(iii)** $4x + z = 9$ **J14/32/Q10**

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27 The points A and B have position vectors given by $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$. The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

Answer: **(ii)** $x + 4y + 7z = 19$ **J15/32/Q10**

28 With respect to the origin O, the position vectors of two points A and B are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B, and $\overrightarrow{AP} = \lambda \overrightarrow{AB}$.

(i) Show that $\overrightarrow{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$. [2]

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ, find the value of λ for which OP bisects the angle AOB. [5]

(iii) When λ has this value, verify that $AP : PB = OA : OB$. [1]

Answer: **(ii)** $\lambda = \frac{3}{8}$ **N11/32/Q7**

29 The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

Answers: (ii) $a = 2$, $b = -2$ **N14/32/Q10**

30 Relative to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 10 \\ 0 \\ 6 \end{pmatrix}.$$

(i) Find angle ABC. [6]

The point D is such that $ABCD$ is a parallelogram.

(ii) Find the position vector of D. [2]

J11/12/Q8

Answers: **(i)** 112.4°; **(ii)** $\begin{pmatrix} 8 \\ 1 \\ 8 \end{pmatrix}$.

31 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = 5\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + 7\mathbf{j} + p\mathbf{k},$$

where p is a constant.

(i) Find the value of p for which angle AOB is 90°. [3]

(ii) In the case where $p = 4$, find the vector which has magnitude 28 and is in the same direction as $\overrightarrow{AB}$. [4]

Answers: **(i)** −8½; **(ii)** −12i + 24j + 8k **N11/12/Q3**

32

Answers: **(i)** 31.0° **(ii)** 9**i** − 12**k**, 4**i** + 6**j** − 12**k** **(iii)** $\dfrac{-5\mathbf{i}+6\mathbf{j}}{\sqrt{61}}$

J12/12/Q8

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33 The position vectors of the points A and B, relative to an origin O, are given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} k \\ -k \\ 2k \end{pmatrix},$$

where k is a constant.

(i) In the case where $k = 2$, calculate angle AOB. [4]

(ii) Find the values of k for which $\overrightarrow{AB}$ is a unit vector. [4]

Answers: **(i)** 24.1°; **(ii)** $k = 1$ or $\dfrac{2}{3}$.

N12/12/Q7

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34 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 3\mathbf{i} + p\mathbf{j} + q\mathbf{k},$$

where p and q are constants.

(i) State the values of p and q for which $\overrightarrow{OA}$ is parallel to $\overrightarrow{OB}$. [2]

(ii) In the case where $q = 2p$, find the value of p for which angle BOA is 90°. [2]

(iii) In the case where $p = 1$ and $q = 8$, find the unit vector in the direction of $\overrightarrow{AB}$. [3]

Answers: **(i)** $p = -6$, $q = 6$. **(ii)** −1.5. **(iii)** $\dfrac{1}{7}$ (2**i** + 3**j** + 6**k**).

J13/12/Q6

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35 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} \quad \text{and} \quad \overrightarrow{OB} = 4\mathbf{i} + p\mathbf{k}.$$

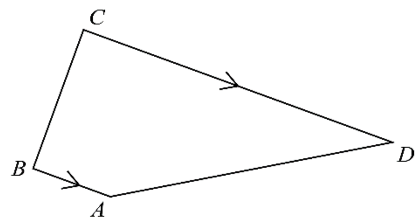
(i) In the case where $p = 6$, find the unit vector in the direction of $\overrightarrow{AB}$. [3]

(ii) Find the values of p for which angle $AOB = \cos^{-1}\left(\frac{1}{5}\right)$. [4]

Answers: **(i)** $\dfrac{1}{7}$ (3**i** – 2**j** + 6**k**). **(ii)** $p = \pm 8$.

N13/12/Q4

36



The diagram shows a trapezium $ABCD$ in which BA is parallel to CD. The position vectors of A, B and C relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

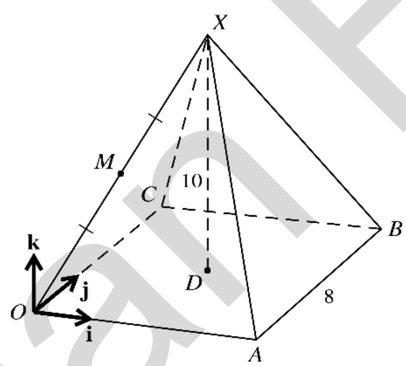
(i) Use a scalar product to show that AB is perpendicular to BC. [3]

(ii) Given that the length of CD is 12 units, find the position vector of D. [4]

Answers: **(i)** Proof **(ii)** $\begin{pmatrix} 12 \\ 9 \\ -2 \end{pmatrix}$

J14/12/Q7

37



The diagram shows a pyramid $OABCX$. The horizontal square base $OABC$ has side 8 units and the centre of the base is D. The top of the pyramid, X, is vertically above D and $XD = 10$ units. The mid-point of OX is M. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively and the unit vector $\mathbf{k}$ is vertically upwards.

(i) Express the vectors $\overrightarrow{AM}$ and $\overrightarrow{AC}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(ii) Use a scalar product to find angle MAC. [4]

Answers: **(i)** $-6\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $-8\mathbf{i} + 8\mathbf{j}$, **(ii)** 45.4°

N14/12/Q7

38 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

(i) Use a vector method to find angle AOB. [4]

The point C is such that $\overrightarrow{AB} = \overrightarrow{BC}$.

(ii) Find the unit vector in the direction of $\overrightarrow{OC}$. [4]

(iii) Show that triangle OAC is isosceles. [1]

Answer: (i) 31.8°; (ii) $\frac{1}{6}(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$

J15/12/Q9

HOMEWORK: VECTORS– VARIANTS 31 & 33

1 Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Show that angle ABC is 90°. [4]

(ii) Find the area of triangle ABC, giving your answer correct to 1 decimal place. [3]

Answer: (ii) 8.6. 13/J15/5

---

2 Relative to the origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}.$$

(i) Find the cosine of angle AOB. [3]

The position vector of C is given by $\overrightarrow{OC} = \begin{pmatrix} k \\ -2k \\ 2k-3 \end{pmatrix}$.

(ii) Given that AB and OC have the same length, find the possible values of k. [4]

Answers: (i) $\frac{2}{7}$ (ii) $3, -\frac{5}{3}$ 11/J15/4

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3 Three points, O, A and B, are such that $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + p\mathbf{k}$ and $\overrightarrow{OB} = -7\mathbf{i} + (1-p)\mathbf{j} + p\mathbf{k}$, where p is a constant.

(i) Find the values of p for which $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$. [3]

(ii) The magnitudes of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are a and b respectively. Find the value of p for which $b^2 = 2a^2$. [2]

(iii) Find the unit vector in the direction of $\overrightarrow{AB}$ when $p = -8$. [3]

Answers: (i) –1, 4; (ii) 15; (iii) 1/5(–4i + 3j) 13/N14/7

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4 Relative to an origin O, the position vector of A is $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and the position vector of B is $7\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

(i) Show that angle OAB is a right angle. [4]

(ii) Find the area of triangle OAB. [3]

Answer: 12.5 (or exact equivalent) 11/N14/6

5 The position vectors of points A, B and C relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$$

(i) Show that angle $BAC = \cos^{-1}(\frac{1}{3})$. [5]

(ii) Use the result in part (i) to find the exact value of the area of triangle ABC. [3]

Answer: (ii) $5\sqrt{8}$. 13/J14/7

6 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3p \\ 4 \\ p^2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} -p \\ -1 \\ p^2 \end{pmatrix}.$$

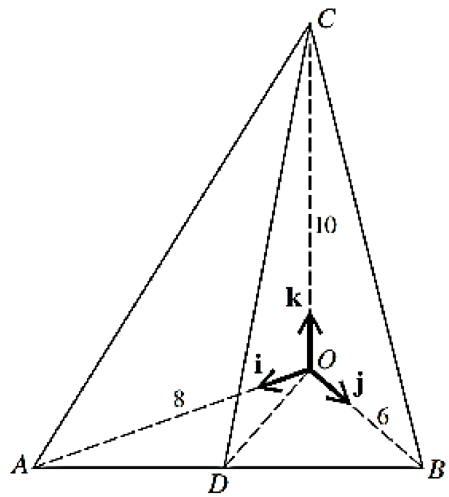
(i) Find the values of p for which angle AOB is 90°. [3]

(ii) For the case where $p = 3$, find the unit vector in the direction of $\overrightarrow{BA}$. [3]

11/J14/8

Answers: (i) $p = \pm 2$; (ii) $\frac{1}{13}\begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}$

7



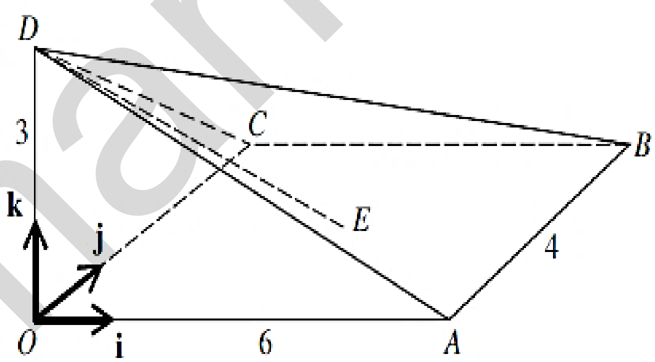
The diagram shows a pyramid $OABC$ in which the edge OC is vertical. The horizontal base OAB is a triangle, right-angled at O, and D is the mid-point of AB. The edges OA, OB and OC have lengths of 8 units, 6 units and 10 units respectively. The unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

(i) Express each of the vectors $\overrightarrow{OD}$ and $\overrightarrow{CD}$ in terms of **i**, **j** and **k**. [2]

(ii) Use a scalar product to find angle ODC. [4]

Answer. 4i + 3j, 4i + 3j – 10k, 63.4 — 13/N13/4

8



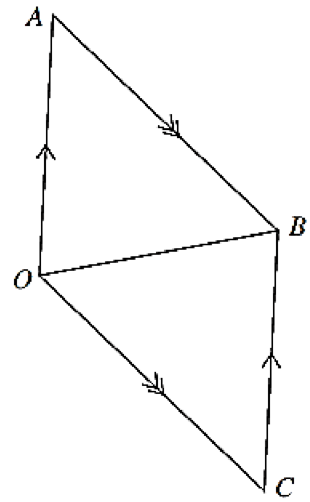
[illegible]

(i) Express each of the vectors $\overrightarrow{DB}$ and $\overrightarrow{DE}$ in terms of **i**, **j** and **k**. [2]

(ii) Use a scalar product to find angle BDE. [4]

Answers: **(i)** DB = 6i + 4j – 3k, DE = 3i + 2j –3k; **(ii)** 17.2°. — 11/N13/3

9



The diagram shows a parallelogram $OABC$ in which

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}.$$

(i) Use a scalar product to find angle BOC. [6]

(ii) Find a vector which has magnitude 35 and is parallel to the vector $\overrightarrow{OC}$. [2]

Answers: **(i)** 54.3° or 0.948 rads.; **(ii)** either 5(2i–3j+6k) or –5(2i–3j+6k). 13/J13/8

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10 Relative to an origin O, the position vectors of three points, A, B and C, are given by

$$\overrightarrow{OA} = \mathbf{i} + 2p\mathbf{j} + q\mathbf{k}, \quad \overrightarrow{OB} = q\mathbf{j} - 2p\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = -(4p^2 + q^2)\mathbf{i} + 2p\mathbf{j} + q\mathbf{k},$$

where p and q are constants.

(i) Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$ for all non-zero values of p and q. [2]

(ii) Find the magnitude of $\overrightarrow{CA}$ in terms of p and q. [2]

(iii) For the case where $p = 3$ and $q = 2$, find the unit vector parallel to $\overrightarrow{BA}$. [3]

Answers: **(i)** OA.OC = $-4p^2 - q^2 + 4p^2 + q^2 = 0$ hence perpendicular; **(ii)** |CA| = $1 + 4p^2 + q^2$; 11/J13/6
(iii) 1/9 (i + 4j +8k).

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11 The position vectors of points A and B relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} p \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ p \end{pmatrix},$$

where p is a constant.

(i) In the case where OAB is a straight line, state the value of p and find the unit vector in the direction of $\overrightarrow{OA}$. [3]

(ii) In the case where OA is perpendicular to AB, find the possible values of p. [5]

(iii) In the case where $p = 3$, the point C is such that $OABC$ is a parallelogram. Find the position vector of C. [2]

13/N12/9

Answers: (i) 2, $\sqrt{6}\begin{pmatrix}2\\1\\1\end{pmatrix}$; (ii) 0 or 5; (iii) $\begin{pmatrix}1\\1\\2\end{pmatrix}$.

---

12 The position vectors of points A and B relative to an origin O are **a** and **b** respectively. The position vectors of points C and D relative to O are 3**a** and 2**b** respectively. It is given that

$$\mathbf{a} = \begin{pmatrix}2\\1\\2\end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix}4\\0\\6\end{pmatrix}.$$

(i) Find the unit vector in the direction of $\overrightarrow{CD}$. [3]

(ii) The point E is the mid-point of CD. Find angle EOD. [6]

Answers: (i) 1/7(2i – 3j +6k); (ii) 8.6°. 11/N12/9

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13 Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix}2\\-1\\4\end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix}4\\2\\-2\end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix}1\\3\\p\end{pmatrix}.$$

Find

(i) the unit vector in the direction of $\overrightarrow{AB}$, [3]

(ii) the value of the constant p for which angle $BOC = 90°$. [2]

13/J12/2

Answers: (i) $\frac{1}{7}\begin{pmatrix}2\\3\\-6\end{pmatrix}$ (ii) p = 5

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14 Two vectors **u** and **v** are such that $\mathbf{u} = \begin{pmatrix}p^2\\-2\\6\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}2\\p-1\\2p+1\end{pmatrix}$, where p is a constant.

(i) Find the values of p for which **u** is perpendicular to **v**. [3]

(ii) For the case where $p = 1$, find the angle between the directions of **u** and **v**. [4]

Answers: (i) – 1, – 4 (ii) 30.0° 11/J12/6

15 Relative to an origin O, the position vectors of points A and B are $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ respectively.

(i) Use a scalar product to find angle BOA. [4]

The point C is the mid-point of AB. The point D is such that $\overrightarrow{OD} = 2\overrightarrow{OB}$.

(ii) Find $\overrightarrow{DC}$. [4]

Answers: (i) 71.4, (ii) $-6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ 13/N11/6

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16 Relative to an origin O, the point A has position vector $4\mathbf{i} + 7\mathbf{j} - p\mathbf{k}$ and the point B has position vector $8\mathbf{i} - \mathbf{j} - p\mathbf{k}$, where p is a constant.

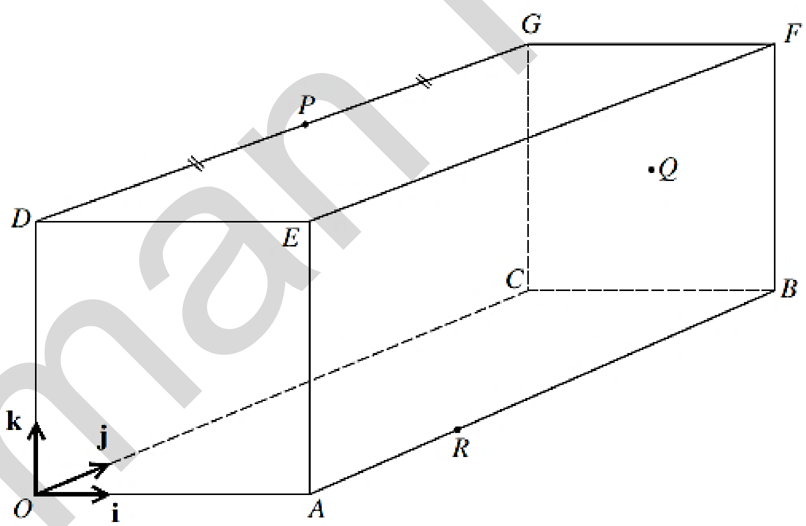
(i) Find $\overrightarrow{OA}.\overrightarrow{OB}$. [2]

(ii) Hence show that there are no real values of p for which OA and OB are perpendicular to each other. [1]

(iii) Find the values of p for which angle $AOB = 60°$. [4]

Answers: (i) $25 + p^2$; (ii) $25 + p^2 = 0$ has no real solutions; (iii) $p = \pm\sqrt{15}$. 11/N11/8

17



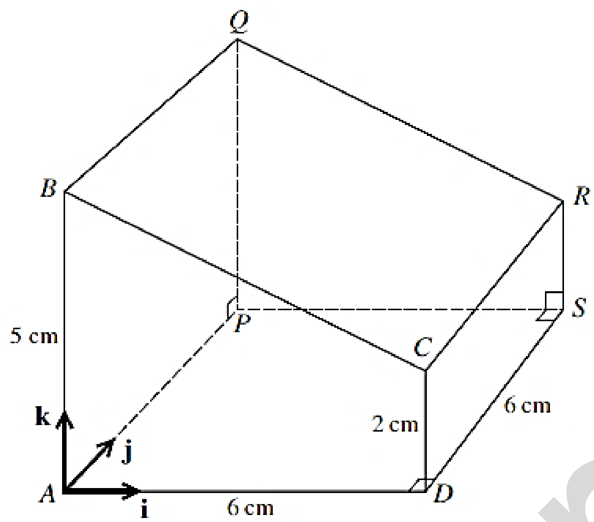
In the diagram, $OABCDEFG$ is a rectangular block in which $OA = OD = 6$ cm and $AB = 12$ cm. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The point P is the mid-point of DG, Q is the centre of the square face $CBFG$ and R lies on AB such that $AR = 4$ cm.

(i) Express each of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

(ii) Use a scalar product to find angle RQP. [4]

Answers: (i) $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, $-3\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$; (ii) 63.2°. 13/J11/5

18



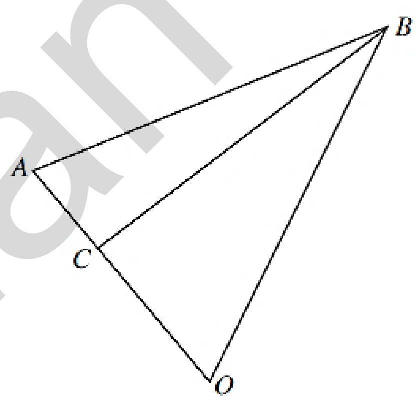
The diagram shows a prism $ABCDPQRS$ with a horizontal square base $APSD$ with sides of length 6 cm. The cross-section $ABCD$ is a trapezium and is such that the vertical edges AB and DC are of lengths 5 cm and 2 cm respectively. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to AD, AP and AB respectively.

(i) Express each of the vectors $\overrightarrow{CP}$ and $\overrightarrow{CQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [2]

(ii) Use a scalar product to calculate angle PCQ. [4]

Answers: (i) $-6\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$, $-6\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$; (ii) 32.7°. 11/J11/4

19



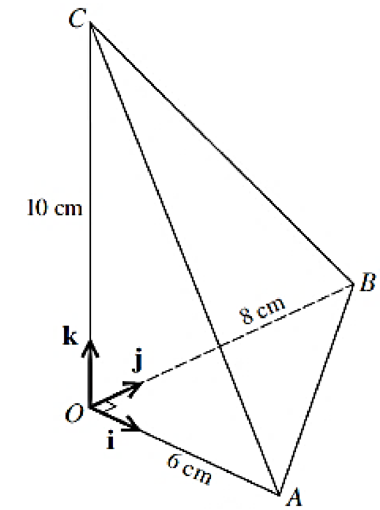
(i) Find angle AOB. [4]

(ii) Find $\overrightarrow{BC}$ in terms of p and vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [1]

(iii) Find the value of p given that BC is perpendicular to OA. [4]

Answers: (i) 66.6°; (ii) $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + p(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$; (iii) $\frac{4}{7}$. 13/N10/10

20



The diagram shows a pyramid $OABC$ with a horizontal base OAB where $OA = 6$ cm, $OB = 8$ cm and angle $AOB = 90°$. The point C is vertically above O and $OC = 10$ cm. Unit vectors **i**, **j** and **k** are parallel to OA, OB and OC as shown.

Use a scalar product to find angle ACB. [6]

Answer: 48.0° 11/N10/5

21 Relative to an origin O, the position vectors of the points A, B and C are given by

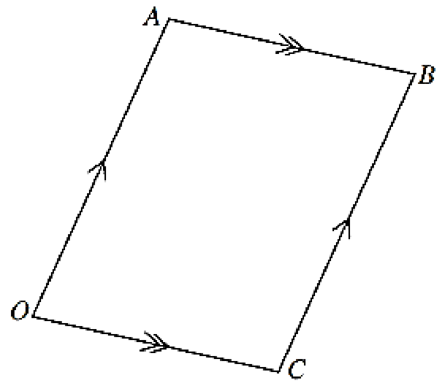
$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}, \quad \overrightarrow{OC} = -\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}.$$

(i) Use a scalar product to find angle ABC. [6]

(ii) Find the perimeter of triangle ABC, giving your answer correct to 2 decimal places. [2]

Answers: **(i)** 63.6°; **(ii)** 18.32. 13/J10/6

22



The diagram shows the parallelogram $OABC$. Given that $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, find

(i) the unit vector in the direction of $\overrightarrow{OB}$, [3]

(ii) the acute angle between the diagonals of the parallelogram, [5]

(iii) the perimeter of the parallelogram, correct to 1 decimal place. [3]

Answers: **(i)** $\frac{1}{6}(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$; **(ii)** 74.2°; **(iii)** 15.4.

11/J10/10

23 The straight line l_1 passes through the points $(0, 1, 5)$ and $(2, -2, 1)$. The straight line l_2 has equation $\mathbf{r} = 7\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$.

(i) Show that the lines l_1 and l_2 are skew. [6]

(ii) Find the acute angle between the direction of the line l_2 and the direction of the x-axis. [3]

Answer: **(ii)** 1.39 radians or 79.5°

31/J15/6

24 The equations of two straight lines are

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = a\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3a\mathbf{k}),$$

where a is a constant.

(i) Show that the lines intersect for all values of a. [4]

(ii) Given that the point of intersection is at a distance of 9 units from the origin, find the possible values of a. [4]

Answer: **(ii)** –2, 3

33/N14/7

25 The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

Answers: **(ii)** $a = 2$, $b = -2$

31/N14/10

26 The lines l and m have equations $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} - \mathbf{k})$ respectively, where a and b are constants.

(i) Given that l and m intersect, show that

$$2a - b = 4. \qquad [4]$$

(ii) Given also that l and m are perpendicular, find the values of a and b. [4]

(iii) When a and b have these values, find the position vector of the point of intersection of l and m. [2]

Answers: (ii) $a = 3$, $b = 2$ (iii) $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ 33/J12/9

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27 The point P has coordinates $(-1, 4, 11)$ and the line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

(i) Find the perpendicular distance from P to l. [4]

Answers: (i) $\sqrt{104}$ (ii) $3x - 9y + z = -28$ 31/J12/8

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28 With respect to the origin O, the position vectors of two points A and B are given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B, and $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

(i) Show that $\overrightarrow{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$. [2]

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ, find the value of λ for which OP bisects the angle AOB. [5]

(iii) When λ has this value, verify that $AP : PB = OA : OB$. [1]

Answer: (ii) $\lambda = \frac{3}{8}$ 31/N11/7

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29 With respect to the origin O, the lines l and m have vector equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

(i) Prove that l and m do not intersect. [4]

(ii) Calculate the acute angle between the directions of l and m. [3]

Answers: (ii) 47.1°; 33/J11/10

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30 With respect to the origin O, the points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line AB and OP is perpendicular to AB.

(i) Find a vector equation for the line AB. [1]

(ii) Find the position vector of P. [4]

Answers: (i) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$; (ii) $\frac{2}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$; (iii) $2x + 5y + 7z = 26$. 31/N10/7

31 The lines l and m have vector equations

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that l and m intersect. **[4]**

(ii) Calculate the acute angle between the lines. **[3]**

Answers: **(ii)** 74.2°; **(iii)** $5x - 3y - 4z = -2$. **31/J10/10**

32 The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

N16/33/Q10

Answers: $-\mathbf{i} + 3\mathbf{j}$ and $\frac{7}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{5}{3}\mathbf{k}$

33 The points A and B have position vectors, relative to the origin O, given by $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{k}$. The line l has vector equation $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

(i) Show that the line passing through A and B does not intersect l. [4]

(ii) Show that the length of the perpendicular from A to l is $\frac{1}{\sqrt{2}}$. [5]

J16/33/Q8

34 The points A and B have position vectors given by $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$, where m is a constant.

(i) Given that the line l intersects the line passing through A and B, find the value of m. [5]

Answers: **(i)** $m = 3$ **(ii)** $2x - y + z = 6$ **J17/33/Q10**

35 The point P has position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(i) Find the length of the perpendicular from P to l, giving your answer correct to 3 significant figures. [5]

Answer: **(i)** 1.22, **(ii)** $4x + y - 2z = 8$ **J18/31/Q10**

36 The points A and B have position vectors $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line l has equation $\mathbf{r} = 4\mathbf{i} + 6\mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

(i) Show that l does not intersect the line passing through A and B. [5]

19

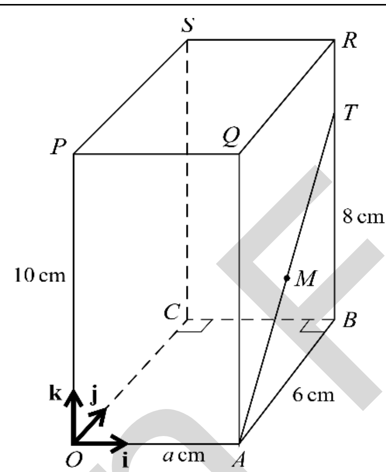
The point P, with parameter t, lies on l and is such that angle PAB is equal to $120°$.

(ii) Show that $3t^2 + 8t + 4 = 0$. Hence find the position vector of P. [6]

Answer: **(ii)** $2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

J18/33/Q10

37



The diagram shows a cuboid $OABCPQRS$ with a horizontal base $OABC$ in which $AB = 6$ cm and $OA = a$ cm, where a is a constant. The height OP of the cuboid is 10 cm. The point T on BR is such that $BT = 8$ cm, and M is the mid-point of AT. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OP respectively.

(i) For the case where $a = 2$, find the unit vector in the direction of $\overrightarrow{PM}$. [4]

(ii) For the case where angle $ATP = \cos^{-1}\left(\frac{2}{7}\right)$, find the value of a. [5]

Answers: **(i)** $\frac{1}{2}(2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})$ **(ii)** 3

N15/11/Q10

38 Relative to an origin O, the position vectors of the points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} p-6 \\ 2p-6 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 4-2p \\ p \\ 2 \end{pmatrix},$$

where p is a constant.

(i) For the case where OA is perpendicular to OB, find the value of p. [3]

(ii) For the case where OAB is a straight line, find the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Find also the length of the line OA. [4]

Answer: (i) $p = 2.2$; (ii) $OA = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $OE = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$, length of OA is 3.

N15/13/Q5

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39 Relative to an origin O, the position vectors of points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 5 \\ -1 \\ k \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$$

respectively, where k is a constant.

(i) Find the value of k in the case where angle $AOB = 90°$. [2]

(ii) Find the possible values of k for which the lengths of AB and OC are equal. [4]

The point D is such that $\overrightarrow{OD}$ is in the same direction as $\overrightarrow{OA}$ and has magnitude 9 units. The point E is such that $\overrightarrow{OE}$ is in the same direction as $\overrightarrow{OC}$ and has magnitude 14 units.

(iii) Find the magnitude of $\overrightarrow{DE}$ in the form $\sqrt{n}$ where n is an integer. [4]

Answers: (i) $k = 4.5$ (ii) $k = 4$, $k = -8$ (iii) $\sqrt{85}$

J16/11/Q10

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40 The position vectors of A, B and C relative to an origin O are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 5 \\ p \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix},$$

where p is a constant.

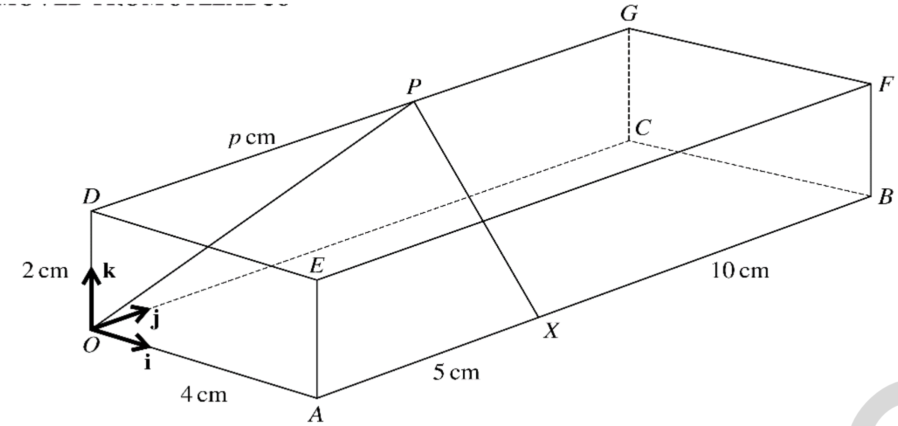
(i) Find the value of p for which the lengths of AB and CB are equal. [4]

(ii) For the case where $p = 1$, use a scalar product to find angle ABC. [4]

Answers: (i) $p = 2$; (ii) 75.3°.

J16/13/Q9

41



The diagram shows a cuboid $OABCDEFG$ with a horizontal base $OABC$ in which $OA = 4$ cm and $AB = 15$ cm. The height OD of the cuboid is 2 cm. The point X on AB is such that $AX = 5$ cm and the point P on DG is such that $DP = p$ cm, where p is a constant. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively.

(i) Find the possible values of p such that angle $OPX = 90°$. [4]

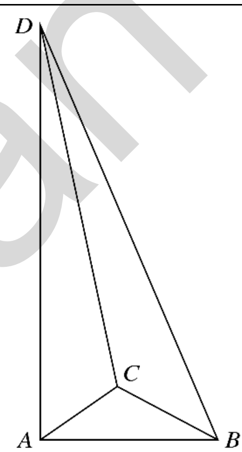
(ii) For the case where $p = 9$, find the unit vector in the direction of $\overrightarrow{XP}$. [2]

(iii) A point Q lies on the face $CBFG$ and is such that XQ is parallel to AG. Find $\overrightarrow{XQ}$. [3]

Answers: **(i)** p □ 1, p □ 4 **(ii)** (1/6)(–4**i** + 4**j** + 2**k**) **(iii)** (2/3)(–4**i** + 15**j** + 2**k**)

N16/11/Q9

42



The diagram shows a triangular pyramid $ABCD$. It is given that

$$\overrightarrow{AB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{AD} = \mathbf{i} + 4\mathbf{j} - 7\mathbf{k}.$$

(i) Verify, showing all necessary working, that each of the angles DAB, DAC and CAB is 90°. [3]

(ii) Find the exact value of the area of the triangle ABC, and hence find the exact value of the volume of the pyramid. [4]

[The volume V of a pyramid of base area A and vertical height h is given by $V = \frac{1}{3}Ah$.]

Answers: **(ii)** Area of triangle $ABC = \frac{1}{2}\sqrt{66}$, Volume of pyramid = 11. **N16/13/Q7**

43 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -6 \\ p \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 2 \\ -6 \\ -7 \end{pmatrix},$$

and angle $AOB = 90°$.

(i) Find the value of p. [2]

The point C is such that $\overrightarrow{OC} = \frac{2}{3}\overrightarrow{OA}$.

(ii) Find the unit vector in the direction of $\overrightarrow{BC}$. [4]

Answers: **(i)** $p = 6$ **(ii)** $\frac{1}{5\sqrt{5}}(2\mathbf{j} + 11\mathbf{k})$ or equivalent **J17/11/Q2**

44 Relative to an origin O, the position vectors of points A and B are given by

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} 5 \\ 4 \\ -3 \end{pmatrix}.$$

The point P lies on AB and is such that $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$.

(i) Find the position vector of P. [3]

(ii) Find the distance OP. [1]

(iii) Determine whether OP is perpendicular to AB. Justify your answer. [2]

Answers : **(i)** $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$; **(ii)** $\sqrt{30}$ or 5.48; **(iii)** Perpendicular. **J17/13/Q4**

45 **(a)** Relative to an origin O, the position vectors of two points P and Q are $\mathbf{p}$ and $\mathbf{q}$ respectively. The point R is such that PQR is a straight line with Q the mid-point of PR. Find the position vector of R in terms of $\mathbf{p}$ and $\mathbf{q}$, simplifying your answer. [3]

(b) The vector $6\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ has magnitude 21 and is perpendicular to $3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Find the possible values of a and b, showing all necessary working. [6]

Answers: **(a)** $2q - p$ **(b)** $a = 9$, $b = -18$ *or* $a = -18$, $b = 9$ **N17/11/Q8**

46 Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 8 \\ -6 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} -10 \\ 3 \\ -13 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}.$$

A fourth point, D, is such that the magnitudes $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ and $|\overrightarrow{CD}|$ are the first, second and third terms respectively of a geometric progression.

(i) Find the magnitudes $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ and $|\overrightarrow{CD}|$. [5]

(ii) Given that D is a point lying on the line through B and C, find the two possible position vectors of the point D. [4]

N17/13/Q9

Answers: **(i)** $|\mathbf{AB}| = 27$, $|\mathbf{BC}| = 18$, $|\mathbf{CD}| = 12$ **(ii)** $\mathbf{OD} = \begin{pmatrix} 10 \\ -7 \\ 7 \end{pmatrix} \begin{pmatrix} -6 \\ 1 \\ -9 \end{pmatrix}$

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47 Relative to an origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.$$

(i) Find $\overrightarrow{AC}$. [1]

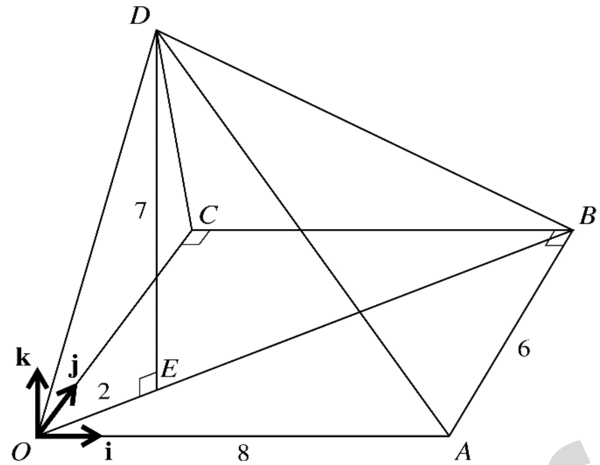
(ii) The point M is the mid-point of AC. Find the unit vector in the direction of $\overrightarrow{OM}$. [3]

(iii) Evaluate $\overrightarrow{AB}.\overrightarrow{AC}$ and hence find angle BAC. [4]

J18/11/Q7

Answers: **(i)** $2i + 4j - 4k$ **(ii)** $\dfrac{1}{\sqrt{5}(2i - j)}$ **(iii)** $79.0°$

48



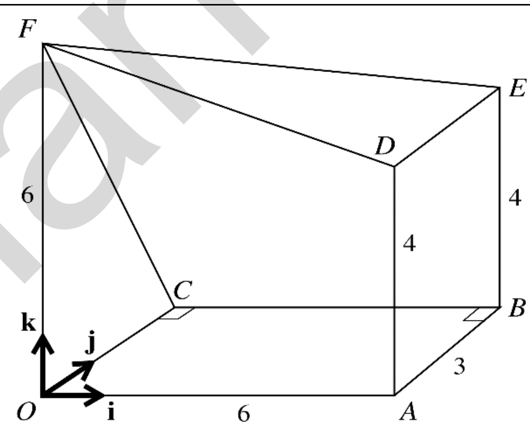
The diagram shows a pyramid $OABCD$ with a horizontal rectangular base $OABC$. The sides OA and AB have lengths of 8 units and 6 units respectively. The point E on OB is such that $OE = 2$ units. The point D of the pyramid is 7 units vertically above E. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and ED respectively.

(i) Show that $\overrightarrow{OE} = 1.6\mathbf{i} + 1.2\mathbf{j}$. [2]

(ii) Use a scalar product to find angle BDO. [7]

Answer: **(ii)** 64.8°. **J18/13/Q9**

49



The diagram shows a solid figure $OABCDEF$ having a horizontal rectangular base $OABC$ with $OA = 6$ units and $AB = 3$ units. The vertical edges OF, AD and BE have lengths 6 units, 4 units and 4 units respectively. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OF respectively.

(i) Find $\overrightarrow{DF}$. [1]

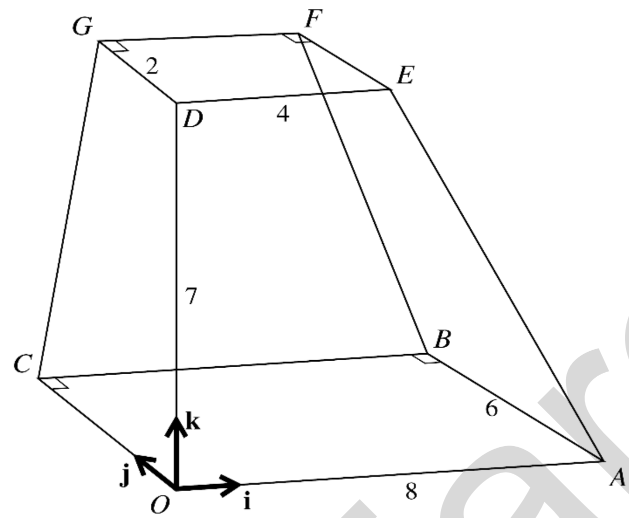
(ii) Find the unit vector in the direction of $\overrightarrow{EF}$. [3]

(iii) Use a scalar product to find angle EFD. [4]

Answers: **(i)** $(-6i+2k)$ **(ii)** $\frac{1}{7}(-6i-3j+2k)$ **(iii)** 25.4^o

N18/11/Q8

50



The diagram shows a solid figure $OABCDEFG$ with a horizontal rectangular base $OABC$ in which $OA = 8$ units and $AB = 6$ units. The rectangle $DEFG$ lies in a horizontal plane and is such that D is 7 units vertically above O and DE is parallel to OA. The sides DE and DG have lengths 4 units and 2 units respectively. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OD respectively. Use a scalar product to find angle OBF, giving your answer in the form $\cos^{-1}\left(\frac{a}{b}\right)$, where a and b are integers.

[6]

Answer: $\cos^{-1}\left(\frac{28}{45}\right)$

N18/13/Q6

3 Pure Mathematics 3 (for Paper 3)

Knowledge of the content of Paper 1: Pure Mathematics 1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.

3.1 Algebra

Candidates should be able to:	Notes and examples
• understand the meaning of $\lvert x \rvert$, sketch the graph of $y = \lvert ax + b \rvert$ and use relations such as $\lvert a \rvert = \lvert b \rvert \Leftrightarrow a^2 = b^2$ and $\lvert x - a \rvert < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities	Graphs of $y = \lvert f(x) \rvert$ and $y = f(\lvert x \rvert)$ for non-linear functions f are not included. e.g. $\lvert 3x - 2 \rvert = \lvert 2x + 7 \rvert$, $2x + 5 < \lvert x + 1 \rvert$.
• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero)	
• use the factor theorem and the remainder theorem	e.g. to find factors and remainders, solve polynomial equations or evaluate unknown coefficients. Including factors of the form $(ax + b)$ in which the coefficient of x is not unity, and including calculation of remainders.
• recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than – $(ax + b)(cx + d)(ex + f)$ – $(ax + b)(cx + d)^2$ – $(ax + b)(cx^2 + d)$	Excluding cases where the degree of the numerator exceeds that of the denominator
• use the expansion of $(1 + x)^n$, where n is a rational number and $\lvert x \rvert < 1$.	Finding the general term in an expansion is not included. Adapting the standard series to expand e.g. $\left(2 - \frac{1}{2}x\right)^{-1}$ is included, and determining the set of values of x for which the expansion is valid in such cases is also included.

3 Pure Mathematics 3

3.2 Logarithmic and exponential functions

Candidates should be able to:	Notes and examples
• understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base)	
• understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs	Including knowledge of the graph of $y = e^{kx}$ for both positive and negative values of k.
• use logarithms to solve equations and inequalities in which the unknown appears in indices	e.g. $2^x < 5$, $3 \times 2^{3x-1} < 5$, $3^{x+1} = 4^{2x-1}$.
• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.	e.g. $y = kx^n$ gives $\ln y = \ln k + n \ln x$ which is linear in $\ln x$ and $\ln y$. $y = k(a^x)$ gives $\ln y = \ln k + x \ln a$ which is linear in x and $\ln y$.

3.3 Trigonometry

Candidates should be able to:	Notes and examples
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude	
• use trigonometrical identities for the simplification and exact evaluation of expressions, and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of – $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$ – the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ – the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$ – the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$.	e.g. simplifying $\cos(x - 30^\circ) - 3\sin(x - 60^\circ)$. e.g. solving $\tan\theta + \cot\theta = 4$, $2\sec^2\theta - \tan\theta = 5$, $3\cos\theta + 2\sin\theta = 1$.

3 Pure Mathematics 3

3.4 Differentiation

Candidates should be able to:	Notes and examples
• use the derivatives of e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\tan^{-1} x$, together with constant multiples, sums, differences and composites	Derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are not required.
• differentiate products and quotients	e.g. $\dfrac{2x-4}{3x+2}$, $x^2 \ln x$, xe^{1-x^2}.
• find and use the first derivative of a function which is defined parametrically or implicitly.	e.g. $x = t - e^{2t}$, $y = t + e^{2t}$. e.g. $x^2 + y^2 = xy + 7$. Including use in problems involving tangents and normals.

3.5 Integration

Candidates should be able to:	Notes and examples
• extend the idea of 'reverse differentiation' to include the integration of e^{ax+b}, $\dfrac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$ and $\dfrac{1}{x^2+a^2}$	Including examples such as $\dfrac{1}{2+3x^2}$.
• use trigonometrical relationships in carrying out integration	e.g. use of double-angle formulae to integrate $\sin^2 x$ or $\cos^2(2x)$.
• integrate rational functions by means of decomposition into partial fractions	Restricted to types of partial fractions as specified in topic 3.1 above.
• recognise an integrand of the form $\dfrac{kf'(x)}{f(x)}$, and integrate such functions	e.g. integration of $\dfrac{x}{x^2+1}$, $\tan x$.
• recognise when an integrand can usefully be regarded as a product, and use integration by parts	e.g. integration of $x\sin 2x$, $x^2 e^{-x}$, $\ln x$, $x\tan^{-1} x$.
• use a given substitution to simplify and evaluate either a definite or an indefinite integral.	e.g. to integrate $\sin^2 2x \cos x$ using the substitution $u = \sin x$.

3 Pure Mathematics 3

3.6 Numerical solution of equations

Candidates should be able to:	Notes and examples
• locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change	e.g. finding a pair of consecutive integers between which a root lies.
• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation	
• understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy.	Knowledge of the condition for convergence is not included, but an understanding that an iteration may fail to converge is expected.

3.7 Vectors

Candidates should be able to:	Notes and examples
• use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$	
• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms	e.g. '$OABC$ is a parallelogram' is equivalent to $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$. The general form of the ratio theorem is not included, but understanding that the midpoint of AB has position vector $\frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$ is expected.
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors	In 2 or 3 dimensions.
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and find the equation of a line, given sufficient information	e.g. finding the equation of a line given the position vector of a point on the line and a direction vector, or the position vectors of two points on the line.
• determine whether two lines are parallel, intersect or are skew, and find the point of intersection of two lines when it exists	Calculation of the shortest distance between two skew lines is not required. Finding the equation of the common perpendicular to two skew lines is also not required.
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points.	e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc. Knowledge of the vector product is not required.

3 Pure Mathematics 3

3.8 Differential equations

Candidates should be able to:	Notes and examples
• formulate a simple statement involving a rate of change as a differential equation	The introduction and evaluation of a constant of proportionality, where necessary, is included.
• find by integration a general form of solution for a first order differential equation in which the variables are separable	Including any of the integration techniques from topic 3.5 above.
• use an initial condition to find a particular solution	
• interpret the solution of a differential equation in the context of a problem being modelled by the equation.	Where a differential equation is used to model a 'real-life' situation, no specialised knowledge of the context will be required.

3.9 Complex numbers

Candidates should be able to:	Notes and examples
• understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal	Notations $\mathrm{Re}\,z$, $\mathrm{Im}\,z$, $\lvert z\rvert$, $\arg z$, z^* should be known. The argument of a complex number will usually refer to an angle θ such that $-\pi < \theta \leqslant \pi$, but in some cases the interval $0 \leqslant \theta < 2\pi$ may be more convenient. Answers may use either interval unless the question specifies otherwise.
• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in Cartesian form $x + \mathrm{i}y$	For calculations involving multiplication or division, full details of the working should be shown.
• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs	e.g. in solving a cubic or quartic equation where one complex root is given.
• represent complex numbers geometrically by means of an Argand diagram	
• carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\mathrm{e}^{\mathrm{i}\theta}$	Including the results $\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and corresponding results for division.
• find the two square roots of a complex number	e.g. the square roots of $5 + 12\mathrm{i}$ in exact Cartesian form. Full details of the working should be shown.
• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers	
• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram	e.g. $\lvert z - a\rvert < k$, $\lvert z - a\rvert = \lvert z - b\rvert$, $\arg(z - a) = \alpha$.

5 List of formulae and statistical tables (MF19)

PURE MATHEMATICS

Mensuration

Volume of sphere = $\frac{4}{3}\pi r^3$

Surface area of sphere = $4\pi r^2$

Volume of cone or pyramid = $\frac{1}{3} \times \text{base area} \times \text{height}$

Area of curved surface of cone = $\pi r \times \text{slant height}$

Arc length of circle $= r\theta$ (θ in radians)

Area of sector of circle $= \frac{1}{2}r^2\theta$ (θ in radians)

Algebra

For the quadratic equation $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d\,, \qquad S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}\,, \qquad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)\,, \qquad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n\text{, where } n \text{ is a positive integer}$$

$$\text{and } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\text{, where } n \text{ is rational and } |x| < 1$$

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1} x \leqslant \tfrac{1}{2}\pi, \qquad 0 \leqslant \cos^{-1} x \leqslant \pi, \qquad -\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi$$

Differentiation

$\mathbf{f}(\boldsymbol{x})$	$\mathbf{f}'(\boldsymbol{x})$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
uv	$v\frac{du}{dx} + u\frac{dv}{dx}$
$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$(n \neq -1)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$	$(\lvert x\rvert < a)$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert$$

Vectors

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

FURTHER PURE MATHEMATICS

Algebra

Summations:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^r}{r!}\mathrm{f}^{(r)}(0) + \ldots$$

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \qquad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 \leqslant x \leqslant 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{(all } x\text{)}$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{2r+1} + \ldots \qquad (-1 < x < 1)$$

Trigonometry

If $t = \tan \tfrac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

Hyperbolic functions

$$\cosh^2 x - \sinh^2 x \equiv 1, \qquad \sinh 2x \equiv 2\sinh x \cosh x, \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad (x \geqslant 1)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad (|x| < 1)$$

Differentiation

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

Integration

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathrm{d}x$	
$\sec x$	$\ln\lvert\sec x+\tan x\rvert = \ln\lvert\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi)\rvert$	$\left(\lvert x\rvert<\tfrac{1}{2}\pi\right)$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan(\tfrac{1}{2}x)\rvert$	$(0<x<\pi)$
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\left(\lvert x\rvert<a\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	$(x>a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	

MECHANICS

Uniformly accelerated motion

$$v = u + at\,, \qquad s = \tfrac{1}{2}(u+v)t\,, \qquad s = ut + \tfrac{1}{2}at^2\,, \qquad v^2 = u^2 + 2as$$

FURTHER MECHANICS

Motion of a projectile

Equation of trajectory is:

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

Elastic strings and springs

$$T = \frac{\lambda x}{l}\,, \qquad E = \frac{\lambda x^2}{2l}$$

Motion in a circle

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \qquad \text{or} \qquad \frac{v^2}{r}$$

Centres of mass of uniform bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere of radius r: $\frac{3}{8}r$ from centre

Hemispherical shell of radius r: $\frac{1}{2}r$ from centre

Circular arc of radius r and angle 2α: $\dfrac{r\sin\alpha}{\alpha}$ from centre

Circular sector of radius r and angle 2α: $\dfrac{2r\sin\alpha}{3\alpha}$ from centre

Solid cone or pyramid of height h: $\frac{3}{4}h$ from vertex

PROBABILITY & STATISTICS

Summary statistics

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad \text{standard deviation} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \qquad \text{standard deviation} = \sqrt{\frac{\Sigma(x-\bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

Discrete random variables

$$\mathrm{E}(X) = \Sigma xp, \qquad \mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2$$

For the binomial distribution $\mathrm{B}(n, p)$:

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \qquad \mu = np, \qquad \sigma^2 = np(1-p)$$

For the geometric distribution Geo(p):

$$p_r = p(1-p)^{r-1}, \qquad \mu = \frac{1}{p}$$

For the Poisson distribution $\mathrm{Po}(\lambda)$

$$p_r = \mathrm{e}^{-\lambda} \frac{\lambda^r}{r!}, \qquad \mu = \lambda, \qquad \sigma^2 = \lambda$$

Continuous random variables

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\,\mathrm{d}x - \{\mathrm{E}(X)\}^2$$

Sampling and testing

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \qquad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Central Limit Theorem:

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$\mathrm{N}\left(p, \frac{p(1-p)}{n}\right)$$

FURTHER PROBABILITY & STATISTICS

Sampling and testing

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

Probability generating functions

$\mathrm{G}_X(t) = \mathrm{E}(t^X)$, $\quad\mathrm{E}(X) = \mathrm{G}'_X(1)$, $\quad\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - \{\mathrm{G}'_X(1)\}^2$

Changes to this syllabus for 2020, 2021 and 2022

The syllabus has been reviewed and revised for first examination in 2020.

Significant additions to the syllabus content are indicated by vertical black lines either side of the text on pages 12–32 of this syllabus. Other changes, including removed content, are listed below.

In addition to reading the syllabus, teachers should refer to the updated specimen papers.

You are strongly advised to read the whole syllabus before planning your teaching programme.

Carry forward from 2019	• Candidates taking AS Level in 2019 can carry forward their result towards the full A Level with the revised syllabus in 2020.
Changes to availability of Mechanics 2 component	• Following consultation with schools and universities, from 2020 the Mechanics 2 component (formerly Paper 5) is no longer available in AS & A Level Mathematics. See below for information about alternative routes. Mechanics 2 content will be assessed at a higher level in the new AS & A Level Further Mathematics (9231) from 2020.
Changes to option routes through the qualification	• From 2020, there are two option routes to a full A Level Mathematics (9709): A Level candidates take Pure Mathematics 1 + Pure Mathematics 3, plus: EITHER Probability & Statistics 1 + Mechanics OR Probability & Statistics 1 + Probability & Statistics 2. • The numbering of assessment components from 2020 is as follows: Probability & Statistics 1 becomes Paper 5 (formerly Paper 6) Probability & Statistics 2 becomes Paper 6 (formerly Paper 7).
Changes to subject content	• Summary of overall changes to subject content The subject content has been updated following consultation, and given decimal numbering. Some topics have been removed or clarified and others added. The Mechanics 2 content has been removed from the syllabus. Notes and examples have been added to clarify the breadth and depth of content. The prior knowledge requirements have been clarified, to state that simple manipulation of surds and graphs of the form $y = kx^n$ are included. Summary of changes to Pure Mathematics 1 content by section • Section 1 Quadratics: linear inequalities content removed. • Section 2 Functions: transformations content added. • Section 3 Coordinate geometry: circles content added. • Vectors: This section has moved from Paper 1 to Paper 3. • Section 7 Differentiation: limits content added. Summary of changes to Pure Mathematics 2 content by section • Section 1 Algebra: content on sketching a modulus graph added. • Section 5 Integration: trapezium rule retained, but formula for it removed from List of formulae (MF19). *continued*

Changes to subject content continued	Summary of changes to Pure Mathematics 3 content by section • Section 1 Algebra: content on sketching a modulus graph added. • Section 4 Differentiation: the derivative of inverse tangent added. • Section 5 Integration: the idea of 'reverse differentiation' added. • Section 5 Integration: trapezium rule removed from Paper 3. • Section 7 Vectors: vector equations of planes removed; vector content moved from Paper 1 to Paper 3. Summary of changes to Mechanics (formerly Mechanics 1) by section • Section 3 Momentum: linear momentum and direct impact added. Summary of changes to Mechanics 2 • The entirety of the Mechanics 2 content has been removed from A Level Mathematics (9709) and will be part of the new AS & A Level Further Mathematics (9231) from 2020. Summary of changes to Probability & Statistics 1 content by section • Notes added on algebra content and probability notation. • Section 4 Discrete random variables: geometric distribution added. Summary of changes to Probability & Statistics 2 content by section • There are no significant changes to this component. Other changes in the syllabus document • Key concepts for the syllabus have been introduced.
Changes to assessment (including changes to specimen papers)	• Changes to duration of examinations The durations of the Paper 1 examination and the Paper 3 examination have increased by 5 minutes to 1 hour 50 minutes each. • Changes to AOs and aims The single assessment objective (AO) has been divided into two AOs. There is no fundamental change in meaning. The weighting of the AOs has been identified by component. The aims have been clarified. *continued*

Changes to assessment (including changes to specimen papers) continued	• Changes to how question papers are presented Questions will no longer remind candidates to show their working, as a new front cover statement is included. The instructions for candidates on the front cover of examination papers have been amended to add: 'You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.' 'If additional space is required, you should use the lined page at the end of the booklet; the question number or numbers must be clearly shown.' All part questions from 2020 are numbered using the labelling (a), (b), (c), whether the parts are dependent or independent. Roman numerals will only be used for labelling further divisions within a part e.g. (a)(i). Questions will not necessarily be in ascending order of tariff. • Changes to the List of formulae and statistical tables The MF9 List of formulae and statistical tables is being replaced from 2020 with a new list, MF19, which combines formulae for AS & A Level Mathematics (9709) and AS & A Level Further Mathematics (9231). The MF19 list includes new formulae for additional topics introduced, and the trapezium rule is removed. AS & A Level Mathematics candidates will not need to use formulae from the sections with 'Further' in the heading. • Changes to the list of mathematical notation The list of mathematical notation that may be used in examinations for this syllabus has been updated and is available on our website at **www.cambridgeinternational.org/9709** The specimen materials have been revised to reflect the new assessment structure and syllabus content and these are available on our website at **www.cambridgeinternational.org** The syllabus and specimen papers use our new name, Cambridge Assessment International Education.

Any textbooks endorsed to support the syllabus for examination from 2020 are suitable for use with this syllabus.